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Inverse Data Envelopment Analysis of Cost Efficiency with a Network Structure in the Presence of Fuzzy Data

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Abstract

The petrochemical industry is one of the most influential industries in the economy of oil-rich countries such as Iran, as well as in the global economy. To make informed managerial decisions in this field, managers must not only evaluate the past performance of organizations and identify sources of inefficiency, but also take existing constraints into account. Subsequently, they should set appropriate targets in line with the macro policies of the relevant industry and participate intelligently in global competition while moving toward the future. Data Envelopment Analysis (DEA) is a powerful retrospective tool, whereas Inverse Data Envelopment Analysis (IDEA) is a forward-looking approach capable of estimating data. The combination of these two powerful tools enables managers to plan flexibly and knowledgeably. On the other hand, in the real world, data are often fuzzy. Therefore, efforts have been made to extend conventional network DEA models to the case of fuzzy data in order to obtain more realistic results, and to investigate and propose novel solution approaches. This paper proposes, based on the fuzzy arithmetic approach, a model that estimates inputs while keeping the overall process efficiency as well as cost efficiency constant, and allows outputs to change according to managerial preferences based on organizational objectives. The study has been conducted for a manufacturing workshop in the oil and petrochemical industry, and the obtained results indicate the high effectiveness and capability of the proposed model.

Keywords: Two-stage process data envelopment analysis, Inverse data envelopment analysis, Cost efficiency, Fuzzy data.

1 | Introduction

In today's world, informed decision-making for future planning by senior managers of organizations is of great importance. One of the key factors influencing effective planning is the analysis of past performance

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and the identification of sources of inefficiency. One of the most efficient tools for measuring organizational performance is Data Envelopment Analysis (DEA). DEA is a non-parametric method based on linear programming that evaluates the efficiency of homogeneous Decision-Making Units (DMUs). In this approach, based on a set of axioms, a Production Possibility Set (PPS) is constructed, whose boundary serves as an approximation of the production function. If a DMU lies on the efficient frontier, it is considered efficient and receives an efficiency score of one; otherwise, its efficiency score is less than one. In such cases, DEA models can be used to project the DMU onto the efficient frontier and determine an appropriate benchmark.

Farrell [1] was the first to propose a model for measuring the efficiency of units with multiple inputs and a single output. Charnes et al. [2] extended his approach to units with multiple inputs and multiple outputs, naming it DEA and introducing the CCR model for evaluating relative efficiency. Subsequently, the BCC model was introduced by Banker et al. [3]. With the expansion of research in this field, various radial and non-radial models were developed, enabling more accurate measurement of the relative efficiency of similar DMUs and clearer identification of inefficiency sources in their performance.

In the real world, DMUs such as governmental and non-governmental organizations including banks, institutions, schools, and others often operate through two-stage or multi-stage processes with serial or parallel structures. Conventional DEA methods are incapable of properly evaluating the efficiency of network-structured DMUs [4], since they treat the production process as a "black box" and focus solely on initial inputs and final outputs, ignoring intermediate products. As a result, inefficiency sources within sub-processes cannot be examined. DEA models developed to evaluate multi-stage processes are referred to as Network Data Envelopment Analysis (NDEA) models. Two-stage DEA models form the foundation of network DEA models. In these models, the output of the first stage is considered as the input to the second stage and is referred to as an intermediate measure. In the second stage, these intermediate measures are consumed to produce final outputs.

Considerable effort has been devoted to applying mathematical models to real-world problems; however, real-world data are not always precise or deterministic. Data may exhibit a fuzzy nature. Fuzzy set theory was first introduced by Zadeh in 1956 [5]. Subsequently, fuzzy numbers were incorporated into DEA, enhancing the applicability of DEA models in industrial contexts. A critical issue in this area is how to solve fuzzy programming problems. Several approaches have been proposed for solving fuzzy DEA models, including the widely used α-cut approach, fuzzy number ranking methods, and the possibility approach.

Kao and Liu [6], as well as Saati et al. [7], applied the α-cut approach to solve fuzzy DEA models. Puri and Yadav [8] proposed a fuzzy DEA model with undesirable outputs, which was also solved using the α-cut method. Later, Kao and Liu [9] presented a two-stage network DEA model for fuzzy data and proposed an efficiency interval using the α-cut approach. Lozano [10] developed a model for evaluating the efficiency of a two-stage process with fuzzy data. Hatami-Marbini et al. [11], by extending the method of Kao and Liu [6], proposed a new approach for estimating the efficiency of two-stage processes under fuzzy data.

One of the most fundamental approaches in this context is the fuzzy arithmetic approach. Wang et al. [12] introduced this approach based on fractional models. Inputs and outputs are represented by triangular fuzzy numbers, and by applying fuzzy arithmetic operations, each fractional model is transformed into three linear fractional models. By solving these linear models, the lower, middle, and upper bounds of fuzzy efficiency are estimated. Bardhwaj et al. [13] addressed shortcomings of Wang's method and proposed a revised model. Hatami-Marbini et al. [11] developed a model for evaluating the efficiency of a two-stage process with fuzzy data based on the fuzzy arithmetic approach and fractioncl efficiency definitions. A non-radial model with a weighted additive perspective for decomposing the efficiency of a two-stage process with fuzzy data in the presence of undesirable intermediate measures was proposed by Saeedi et al. [14], [15].

Cost efficiency is one of the fundamental concepts in DEA and evaluates the ability of DMUs to use resources optimally relative to their prices. This measure indicates the extent to which a unit can reduce its costs while maintaining the same level of outputs. In fact, cost efficiency is a combination of technical efficiency and allocative efficiency, and it reflects whether a unit not only uses inputs effectively but also employs the optimal input mix consistent with their prices. Therefore, cost efficiency analysis is an important tool for managers and decision-makers to allocate resources more efficiently and identify and eliminate unnecessary costs. The concept of cost efficiency was first introduced by Farrell [1], who decomposed efficiency into three components: technical efficiency, allocative efficiency, and cost efficiency, where cost efficiency is the product of the first two. Farrell's idea was that a DMU should not only extract the maximum possible output from its inputs (technical efficiency) but also select the least-cost combination of inputs (allocative efficiency).

On the other hand, Inverse Data Envelopment Analysis (IDEA) is a forward-looking extension of DEA that estimates future activities following a potential change (or disruption) while preserving the current efficiency level. In some cases, managers due to resource constraints or investment considerations may wish to change input or output levels while maintaining overall process efficiency. Consequently, two key questions arise:

I. If the inputs of a DMU increase within a group of DMUs and its current efficiency score is to be maintained relative to others, how much additional output must be produced? (Estimation of potential outputs).

II. If the outputs of a DMU need to increase to a specified level while maintaining its efficiency score, how much additional input must be provided? (Estimation of potential inputs).

IDEA seeks to answer these questions. DEA is a powerful tool for investment analysis and resource allocation, particularly when managers face resource constraints, and it supports effective decision-making. Wei et al. [16] were the first to propose an IDEA model for estimating inputs (or outputs). Subsequently, Yan et al. [17], Jahanshahloo [18], [19], Hadi-Vencheh [20], and others conducted extensive research in this area. Shiri et al. [21], [22] applied IDEA in a two-stage network framework with a multiplicative perspective to evaluate cost efficiency and proposed their own model. The main contributions of this study are as follows:

I. Development of a novel DEA model for calculating technical efficiency and cost efficiency in the presence of fuzzy data.

II. Application of the fuzzy arithmetic approach to solve the proposed model by converting the fuzzy problem into three linear programming models corresponding to lower, middle, and upper bounds.

III. Avoidance of data manipulation for defuzzification purposes.

IV. Enhanced transparency of data.

V. Presentation of linear programming models with low computational cost.

VI. Estimation of outputs under potential output changes using an IDEA model.

VII. Incorporation of managerial preferences through appropriate weighting schemes in output estimation.

VIII. Implementation and application of the proposed model in the important oil and petrochemical industry.

The remainder of this paper is organized as follows: Section 2 explains independent and relational network models. Section 3 introduces background concepts related to fuzzy data and computations. The definition of cost efficiency and basic models for its calculation are presented in Section 4. Section 5 discusses the importance of IDEA and input estimation using basic models. Section 6 presents the proposed approach. This study proposes a model that not only computes technical efficiency and cost efficiency scores of a two-stage network considering intermediate measures and fuzzy inputs and outputs at lower, middle, and upper bounds using three linear programming models but also estimates output levels by incorporating managerial preferences in input levels. The proposed model is successfully implemented using data collected from a manufacturing workshop in the oil and petrochemical equipment industry, and the results demonstrate the effectiveness of the model, which are discussed in Section 7.

2 | Efficiency Model of Data Envelopment Analysis in a Two-Stage Process

Suppose there exists n DMUs with a two-stage serial structure. In the first sub-process, the inputs $x_{ij}, (i = 1, \dots, m)$ are consumed to produce certain outputs $z_{pj}, (p = 1, \dots, q)$, which are called intermediate measures. Then, in the second sub-process, these intermediate measures are used as inputs to generate the final outputs $y_{rj}, (r = 1, \dots, s)$.

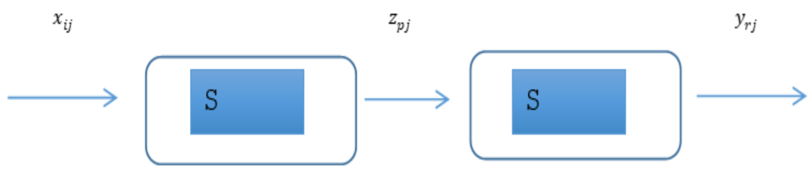

Fig. 1. Structure of a two-stage process in a series configuration.

2.1 | Conventional Data Envelopment Analysis Efficiency Model in a Two-Stage Process

Conventional DEA models (the independent perspective), in evaluating the overall efficiency of a two-stage process, ignore intermediate measures and are therefore unable to examine the sources of inefficiency. Charnes et al. [2] proposed the following model:

$$
\begin{aligned}
& E_o = \max \sum_{r=1}^{s} u_r y_{ro} \\
& \text{s.t.} \quad \sum_{i=1}^{m} v_i x_{io} = 1 \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, j = 1, \dots, n \\
& v_i, , u_r \geq 0, \qquad i = 1, \dots, m , \ r = 1, \dots, s.
\end{aligned} \tag{1}
$$

The optimal solution of *Model (1)* is E_o^*, v_i^*, u_r^*.

2.2 | Relational Data Envelopment Analysis Efficiency Model in a Two-Stage Process

In the relational (network) perspective, intermediate measures are taken into account when evaluating overall efficiency. The model for assessing the overall efficiency of a two-stage process under the assumption of Constant Returns to Scale (CRS) technology was proposed by Kao and Hwang [23]:

$$
\begin{aligned}
& E_o = \max \sum_{r=1}^{s} u_r y_{ro} \\
& \text{s.t.} \quad \sum_{i=1}^{m} v_i x_{io} = 1
\end{aligned} \tag{2}
$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0,\ j = 1, \dots, n$$

$$\sum_{p=1}^{q} w_p z_{pj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, j = 1, \dots, n \quad (2)$$

$$\sum_{p=1}^{q} w_p z_{pj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j = 1, \dots, n$$

$$v_i, w_p, u_r \ge 0, \qquad i = 1, \dots, m,\ p = 1, \dots, q,\ r = 1, \dots, s.$$

The envelopment form of *Model (2)* is given as follows:

$$\min \quad \theta$$

$$\text{s.t.} \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{ij},\ i = 1, \dots, m$$

$$\sum_{j=1}^{n} \lambda_j z_{pj} \ge z_{po},\ i = 1, \dots, m$$

$$\sum_{j=1}^{n} \mu_j z_{pj} \le z_{po},\ i = 1, \dots, m \quad (3)$$

$$\sum_{j=1}^{n} \mu_j y_{rj} \ge y_{ro},\ r = 1, \dots, s$$

$$\lambda_j \ge 0, \mu_j \ge 0,\ j = 1, \dots, n.$$

3 | Fuzzy Data

In real-world situations, input and output data are not always precise and are usually vague and imprecise. One of the major challenges in classical DEA models is the incorporation of fuzzy data.

3.1 | Basic Concepts and Preliminary Definitions of Fuzzy Numbers

Definition 1 ([5], [24]). A triangular fuzzy number $\tilde{A}$ is represented as $\tilde{A} = (a^l, a^m, a^u)$, and its membership function is defined as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a^l}{a^m - a^l}, & a^l \le x \le a^m \\ \dfrac{a^u - x}{a^u - a^m}, & a^m \le x \le a^u \end{cases} \quad (4)$$

Definition 2. A triangular fuzzy number $\tilde{a} = (a^l, a^m, a^u)$ is called positive if $a^l > 0$ [25].

Definition 3. Comparison of two positive triangular fuzzy numbers $\tilde{a} = (a^l, a^m, a^u)$ and $\tilde{b} = (b^l, b^m, b^u)$.

$$\tilde{a} \le \tilde{b} \rightarrow (a^l \le b^l, a^m \le b^m, a^u \le b^u)$$

The arithmetic operations on two positive triangular fuzzy numbers $\tilde{a} = (a^l, a^m, a^u)$ and $\tilde{B} = (b^l, b^m, b^u)$are defined as follows [26], [27]:

$$\begin{aligned}
&\tilde{A} \oplus \tilde{B} = \left(a^l + b^l,\ \ a^m + b^m,\ \ a^u + b^u\right),\\
&\tilde{A} \ominus \tilde{B} = \left(a^l - b^u,\ \ a^m - b^m,\ \ a^u - b^l\right),\\
&\tilde{A} \otimes \tilde{B} = \left(a^l \times b^l,\ \ a^m \times b^m,\ \ a^u \times b^u\right),\\
&k\tilde{A} = \begin{cases} \left(ka^l,\ \ ka^m,\ \ ka^u\right) & k > 0 \\ \left(ka^u,\ \ ka^m,\ \ ka^l\right) & k < 0 \end{cases}, k \in \mathbb{R}\\
&\frac{\tilde{A}}{\tilde{B}} = \frac{\left(a^l, a^m, a^u\right)}{\left(b^l, b^m, b^u\right)} = \left(\frac{a^l}{b^u},\ \ \frac{a^m}{b^m},\ \ \frac{a^u}{b^l}\right), \tilde{A},\ \ \tilde{B} > \tilde{0}.
\end{aligned} \tag{5}$$

4 | Cost Efficiency

Suppose that $c_i \in \mathbb{R}, (i = 1, \dots, m)$ denotes the cost (weight) corresponding to the inputs $x_i, (i = 1, \dots, m)$ and $c^t x^* = \sum_{i=1}^{m} c_i x_{io}^*$ represents the actual observed cost of the DMUs under evaluation. In order to obtain the cost efficiency of the DMU under assessment, the following model is first solved [28], [29]:

$$\begin{aligned}
&c^t X^* = \min \sum_{i=1}^{m} c_i x_i\\
&\text{s.t.} \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{io}, i = 1,\ \dots,\ m,\\
&\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro},\ r = 1,\ \dots,\ s,\\
&\lambda_j \ge 0,\ j = 1,\ \dots, n.
\end{aligned} \tag{6}$$

The optimal solution is (x^*, λ^*). In fact, the objective of solving the above model is to find the minimum levels of x_i that can produce at least the same level of output. Therefore, $c^t X^* = \sum_{i=1}^{m} c_i x_i^*$ represents the minimum calculated cost obtained from the model, and the cost efficiency is defined as follows:

$$CE_o = \frac{c^t X^*}{c^t X_o} = \frac{\sum_{i=1}^{m} c_i x_i^*}{\sum_{i=1}^{m} c_i x_{io}} \tag{7}$$

Definition 4. A DMUs is said to be cost efficient if its cost efficiency score is equal to one; otherwise, its cost efficiency score will be less than one.

4.1 | Cost Efficiency Model of Data Envelopment Analysis in a Two-Stage Process

The cost efficiency enveloping model proposed by Saati et al. [7] can be written as follows:

$$\begin{aligned}
&c^t x^* = \min \sum_{i=1}^{m} c_i x_i\\
&\text{s.t.} \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{io},\ i = 1,\ \dots,\ m,\\
&\sum_{j=1}^{n} (\lambda_j - \mu_j) z_{pj} \ge z_{po},\ p = 1, \dots, q,
\end{aligned} \tag{8}$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro},\ r = 1, \dots, s,$$
$$\lambda_j \geq 0, \quad \mu_j \geq 0,\ j = 1, \dots, n. \tag{8}$$

and if the optimal solution (x^*, λ^*, μ^*) obtained from the above model is, then:

$$CE_o = \frac{c^t x^*}{c^t x_o} = \frac{\sum_{i=1}^{m} c_i x_i^*}{\sum_{i=1}^{m} c_i x_{io}} \tag{9}$$

5 | Inverse Data Envelopment Analysis

Suppose that the output of the unit under evaluation, DMU_O, changes from y_{ro} to $\beta_{ro} = y_{ro} + \beta_{ro}$.To what extent should the input $\alpha_{io} = x_{io} + \Delta x_{io}$ be adjusted? IDEA addresses this question. For the first time, Wei et al. [16] proposed the following model:

$$\begin{aligned} &\min\ (\alpha_1, \alpha_2, \dots, \alpha_m) \\ &\text{s.t.} \sum_{j=1}^{n} \lambda_j x_{ij} \leq \alpha_{io}, i = 1, \dots, m, \\ &\sum_{j=1}^{n} \lambda_j y_{rj} \geq \beta_{ro},\ r = 1, \dots, s, \\ &\lambda_j \geq 0, j = 1, \dots, n. \end{aligned} \tag{10}$$

The above models are multi-objective models, for which various solution methods exist; however, a simple and commonly used approach is the weighted sum method.

$$w_1\alpha_1 + w_2\alpha_2 + \cdots + w_m\alpha_m = \sum_{i=1}^{m} w_i\alpha_i \tag{11}$$

Of course, the weights can be determined based on the manager's preferences, such that $\sum_{i=1}^{m} w_i = 1$.

6 | Proposed Approach

6.1 | Relational Network Data Envelopment Analysis Efficiency Model for a Two-Stage Network (Input-Oriented) with Fuzzy Data

The input-oriented two-stage network envelopment model proposed by Kao and Hwang [23] is extended to the case of fuzzy data and is formulated as *Model (12)*.

$$\begin{aligned} &\min \tilde{\theta} \\ &\text{s.t.} \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \leq \tilde{\theta}\tilde{x}_{ij},\ i = 1, \dots, m, \\ &\sum_{j=1}^{n} \lambda_j \tilde{z}_{pj} \geq \tilde{z}_{pj},\ p = 1, \dots, q, \\ &\sum_{j=1}^{n} \mu_j \tilde{z}_{pj} \leq \tilde{z}_{pj},\ p = 1, \dots, q, \end{aligned} \tag{12}$$

$$\sum_{j=1}^{n} \mu_j \tilde{y}_{rj} \geq \tilde{y}_{ro}, \ r = 1, \dots, s, \tag{12}$$
$$\lambda_j \geq 0, \mu_j \geq 0, \ j = 1, \dots, n.$$

Suppose that $(\tilde{\theta}^*, \lambda^* \mu^*)$ is the optimal solution of the above model. Assume that the fuzzy data, including inputs, outputs, and intermediate measures, are represented by triangular fuzzy numbers.

$$\tilde{x}_{ij} = (x_{ij}^l, x_{ij}^m, x_{ij}^u), \ \tilde{y}_{rjk} = (y_{rj}^l, y_{rj}^m, y_{rj}^u), \tilde{z}_{ijk} = (z_{pj}^l, z_{pj}^m, z_{pj}^u) \tag{13}$$

$$\begin{aligned} \theta_o^{l^*} &= \min \ \theta_o^l \\ \text{s.t.} \ & \sum_{j=1}^{n} \lambda_j x_{ij}^l \leq \theta_o^l x_{io}^u, && i = 1, \dots, m, \\ & \sum_{j=1}^{n} (\lambda_j - \mu_j) z_{pj}^l \geq 0, && p = 1, \dots, q, \\ & \sum_{j=1}^{n} \mu_j y_{rj}^l \geq y_{ro}^l, && r = 1, \dots, s, \\ & \theta_o^l \geq 0, \ \ \lambda_j \geq 0, \ \mu_j \geq 0, && j = 1, \dots, n. \end{aligned} \tag{14}$$

$$\begin{aligned} \theta_o^{m^*} &= \min \theta_o^m \\ \text{s.t.} & \sum_{j=1}^{n} \lambda_j x_{ij}^m \leq \theta_o^m x_{io}^m, i = 1, \ \dots, \ m, \\ & \sum_{j=1}^{n} (\lambda_j - \mu_j) z_{pj}^m \geq 0, \quad p = 1, \dots, q, \\ & \sum_{j=1}^{n} \mu_j y_{rj}^m \geq y_{ro}^m, r = 1, \ \dots, \ s, \\ & \theta_o^m \geq 0, \ \lambda_j \geq 0, \ \mu_j \geq 0, \quad j = 1, \ \dots, \ n. \end{aligned} \tag{15}$$

$$\begin{aligned} \theta_o^{u^*} &= \min \theta_o^u \\ \text{s.t.} & \sum_{j=1}^{n} \lambda_j x_{ij}^u \leq \theta_o^u x_{io}^l, i = 1, \ \dots, \ m, \\ & \sum_{j=1}^{n} (\lambda_j - \mu_j) z_{pj}^u \geq 0, p = 1, \dots, q, \\ & \sum_{j=1}^{n} \mu_j y_{rj}^u \geq y_{ro}^u, r = 1, \ \dots, \ s, \\ & \theta_o^u \geq 0, \ \lambda_j \geq 0, \ \mu_j \geq 0, \quad j = 1, \ \dots, \ n. \end{aligned} \tag{16}$$

Definition 6 ([14], [15]).

I. If $\tilde{\theta} = (\theta^l, \theta^m, \theta^u) = \tilde{1}$ and $\theta^l = \theta^m = \theta^u = \tilde{1}$, then the DMU is strongly efficient.

II. If $\theta^l \leq 1, \theta^m, \theta^u = 1$, then the DMUs is efficient.

III. If $\theta^l, \theta^m \leq 1$and $\theta^u = 1$, then the DMUs is weakly efficient.

IV. If $\theta^l \leq 1, \ \theta^m \leq 1, \ \theta^u \leq 1$, then the DMUs is inefficient.

2.6 | Cost Efficiency Model of Data Envelopment Analysis in a Two-Stage Process with Fuzzy Data

The cost efficiency envelopment model proposed by Shiri et al. [21], [22] can be extended to the case of fuzzy data and written as follows:

$$
\begin{aligned}
& c^t\tilde{x}^* = \min \sum_{i=1}^{m} c_i\tilde{x}_i \\
& \text{s.t.} \ \sum_{j=1}^{n} \lambda_j\tilde{x}_{ij} \le \tilde{x}_{io}, \quad i = 1, \dots, m, \\
& \sum_{j=1}^{n} \lambda_j\tilde{z}_{pj} \ge \tilde{z}_{pj}, \quad p = 1, \dots, q, \\
& \sum_{j=1}^{n} \mu_j\tilde{z}_{pj} \ge \tilde{z}_{pj}, \quad p = 1, \dots, q, \\
& \sum_{j=1}^{n} \mu_j\tilde{y}_{rj} \ge \tilde{y}_{ro}, \quad r = 1, \dots, s, \\
& \lambda_j \ge 0, \mu_j \ge 0, \quad j = 1, \dots, n.
\end{aligned} \tag{17}
$$

$$
CE_o = \frac{c^t\tilde{x}^*}{c^t\tilde{x}_o} = \frac{\sum_{i=1}^{m} c_i\tilde{x}_i^*}{\sum_{i=1}^{m} c_i\tilde{x}_{io}} \tag{18}
$$

According to the triangular representation of the data *Eq. (13)*, we have:

$$
\begin{aligned}
& c^t\tilde{x}^* = \min \left(\sum_{i=1}^{m} c_i x_i^l, \sum_{i=1}^{m} c_i x_i^m, \sum_{i=1}^{m} c_i x_i^u\right) \\
& \text{s.t.} \ \left(\sum_{j=1}^{n} \lambda_j x_{ij}^l, \sum_{j=1}^{n} \lambda_j x_{ij}^m, \sum_{j=1}^{n} \lambda_j x_{ij}^u\right) \le \left(x_i^l, x_i^m, x_i^u\right), \ i = 1, \dots, m, \\
& \left(\sum_{j=1}^{n} (\lambda_j - \mu_j) z_{kj}^l, \sum_{j=1}^{n} (\lambda_j - \mu_j) z_{kj}^m, \sum_{j=1}^{n} (\lambda_j - \mu_j) z_{kj}^u\right) \ge \left(0^l, 0^m, 0^u\right), \ k = 1, \dots, h, \\
& \left(\sum_{j=1}^{n} \mu_j y_{rj}^l, \sum_{j=1}^{n} \mu_j y_{rj}^m, \sum_{j=1}^{n} \mu_j y_{rj}^u\right) \ge \left(y_{ro}^l, y_{ro}^m, y_{ro}^u\right), \ r = 1, \dots, s, \\
& x_i^l, \ x_i^m - x_i^l \ge 0, \ x_i^u - x_i^m \ge 0, \ i = 1, \dots, m, \\
& \lambda_j \ge 0, \ \mu_j \ge 0, \ j = 1, \dots, n.
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
CE_o &= \frac{c^t\tilde{x}^*}{c^t\tilde{x}_o} = \frac{\sum_{i=1}^{m} c_i\tilde{x}_i^*}{\sum_{i=1}^{m} c_i\tilde{x}_{io}} = \frac{\left(\sum_{i=1}^{m} c_i x_i^{l*}, \ \sum_{i=1}^{m} c_i x_i^{m*}, \ \sum_{i=1}^{m} c_i x_i^{u*}\right)}{\left(\sum_{i=1}^{m} c_i x_{io}^{l}, \ \sum_{i=1}^{m} c_i x_{io}^{m}, \ \sum_{i=1}^{m} c_i x_{io}^{u}\right)} = \\
& \left(\frac{\sum_{i=1}^{m} c_i x_i^{l*}}{\sum_{i=1}^{m} c_i x_{io}^{u}}, \frac{\sum_{i=1}^{m} c_i x_i^{m*}}{\sum_{i=1}^{m} c_i x_{io}^{m}}, \frac{\sum_{i=1}^{m} c_i x_i^{u*}}{\sum_{i=1}^{m} c_i x_{io}^{l}}\right) = \left(CE_o^l, CE_o^m, CE_o^u\right).
\end{aligned} \tag{20}
$$

Based on the fuzzy arithmetic approach, three bounds upper, middle, and lower can be obtained for cost efficiency.

$$
\begin{aligned}
& c^t X^{l^*} = \min \quad \sum_{i=1}^{m} c_i x_i^l \\
& \text{s.t.} \quad \sum_{j=1}^{n} \lambda_j x_{ij}^l \le x_i^l, i = 1,\dots,m, \\
& \sum_{j=1}^{n} \left(\lambda_j - \mu_j\right) z_{pj}^u \ge 0, \quad k = 1,\dots,h, \\
& \sum_{j=1}^{n} \mu_j y_{rj}^u \ge y_{ro}^u, \quad r = 1,\dots,s, \\
& x_i^l \ge 0, i = 1,\dots,m, \\
& \lambda_j, \ \mu_j \ge 0, \ \ j = 1,\dots,n.
\end{aligned}
\tag{21}
$$

$$
CE_o^l = \frac{\sum_{i=1}^{m} c_i x_i^{l*}}{\sum_{i=1}^{m} c_i x_{io}^u}
\tag{22}
$$

$$
\begin{aligned}
& c^t X^{m*} = \min \ \sum_{i=1}^{m} c_i x_i^m \\
& \text{s.t.} \quad \sum_{j=1}^{n} \lambda_j x_{ij}^m \le x_i^m, \qquad i = 1,\dots,m, \\
& \sum_{j=1}^{n} (\lambda_j - \mu_j) z_{pj}^m \ge 0^m, \ k = 1,\dots,h, \\
& \sum_{j=1}^{n} \mu_j y_{rj}^m \ge y_{ro}^m, \qquad r = 1,\dots,s, \\
& x_i^m \ge 0, i = 1,\dots,m, \\
& \lambda_j, \ \mu_j \ge 0, \qquad j = 1,\dots,n.
\end{aligned}
\tag{23}
$$

$$
CE_o^m = \frac{\sum_{i=1}^{m} c_i x_i^{m*}}{\sum_{i=1}^{m} c_i x_{io}^m}
\tag{24}
$$

$$
\begin{aligned}
& c^t X^{u*} = \min \ \sum_{i=1}^{m} c_i x_i^u \\
& \text{s.t.} \quad \sum_{j=1}^{n} \lambda_j x_{ij}^u \le x_i^u, \qquad i = 1,\dots,m, \\
& \sum_{j=1}^{n} (\lambda_j - \mu_j) z_{pj}^l \ge 0, \quad k = 1,\dots,h, \\
& \sum_{j=1}^{n} \mu_j y_{rj}^l \ge y_{ro}^l, \qquad r = 1,\dots,s, \\
& x_i^u \ge 0, \ i = 1,\dots,m, \\
& \lambda_j, \ \mu_j \ge 0, \qquad j = 1,\dots,n.
\end{aligned}
\tag{25}
$$

$$
CE_o^u = \frac{\sum_{i=1}^{m} c_i x_i^{u*}}{\sum_{i=1}^{m} c_i x_{io}^l}
\tag{26}
$$

3.6 | Estimation of Inputs Using the Cost Efficiency Inverse Data Envelopment Analysis Model in a Two-Stage Process with Fuzzy Data

Suppose that the outputs of the DMU under evaluation increase from $\tilde{y}_{ro}, (r = 1, \dots, s)$ to $(\tilde{y}_{ro} + \widetilde{\Delta y}_{ro}) = \tilde{\beta}_{ro}$. In order to estimate the amount of change in the input vector, i.e., $\tilde{\alpha} = (\tilde{x}_i + \widetilde{\Delta x}_i)$, $i = 1, \dots, m$, while keeping both the overall process efficiency score and the cost efficiency unchanged, the models presented below must be solved. First, the cost efficiency is calculated by applying the changes in the outputs using *Model (27)*.

$$
\begin{aligned}
& c^t\overline{X} = \min \sum_{i=1}^{m} c_i \tilde{x}_i \\
& \text{s.t.} \ \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \le \tilde{x}_{io}, \quad i = 1, \dots, m, \\
& \sum_{j=1}^{n} \lambda_j \tilde{z}_{pj} \ge \tilde{z}_{pj}, \quad p = 1, \dots, q, \\
& \sum_{j=1}^{n} \mu_j \tilde{z}_{pj} \ge \tilde{z}_{pj}, \quad p = 1, \dots, q, \\
& \sum_{j=1}^{n} \mu_j \tilde{y}_{rj} \ge (\tilde{y}_{ro} + \widetilde{\Delta y}_{ro}) = \tilde{\beta}_{ro}, \ r = 1, \dots, s, \\
& \lambda_j \ge 0, \ \mu_j \ge 0, \quad j = 1, \dots, n.
\end{aligned}
\tag{27}
$$

Subsequently, the following model can be used to estimate the new levels of inputs.

$$
\begin{aligned}
& \min(\tilde{\alpha}_{1o}, \dots, \tilde{\alpha}_{mo}) \\
& \text{s.t.} \ \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \le \tilde{\alpha}_{io}, \ i = 1, \dots, m, \\
& \sum_{j=1}^{n} \lambda_j \tilde{z}_{pj} \ge \tilde{z}_{pj}, \ p = 1, \dots, q, \\
& \sum_{j=1}^{n} \mu_j \tilde{z}_{pj} \ge \tilde{z}_{pj}, \ p = 1, \dots, q, \\
& \sum_{j=1}^{n} \mu_j \tilde{y}_{rj} \ge \tilde{y}_{ro}, \ r = 1, \dots, s, \\
& c^t\tilde{\alpha} = \frac{C^t\bar{x}}{\widetilde{CE}} \\
& \lambda_j \ge 0, \ \mu_j \ge 0, \ j = 1, \dots, n.
\end{aligned}
\tag{28}
$$

Considering *Relation (13)*, the triangular representation of the data can be written as follows:

$$\begin{aligned}
&\min \left([\alpha_1^l,\alpha_1^m,\alpha_1^u],\dots,[\alpha_m^l,\alpha_m^m,\alpha_m^u]\right)\\
&\text{s.t. } \left(\sum_{j=1}^n \lambda_j x_{ij}^l, j\sum_{j=1}^n \lambda_j x_{ij}^m, \sum_{j=1}^n \lambda_j x_{ij}^u\right) \le \left(\theta^{*l}_{input},\theta^{*m}_{input},\theta^{*u}_{input}\right)\left(\alpha_{io}^l,\alpha_{io}^m,\alpha_{io}^u\right), \quad i=1,\dots,m,\\
&\left(\sum_{j=1}^n \lambda_j z_{pj}^l, j\sum_{j=1}^n \lambda_j z_{pj}^m, \sum_{j=1}^n \lambda_j z_{pj}^u\right) \le \left(z_{po}^l,z_{po}^m,z_{po}^u\right), \quad p=1,\dots,q,\\
&\left(\sum_{j=1}^n \mu_j y_{rj}^l, \sum_{j=1}^n \mu_j y_{rj}^m, \sum_{j=1}^n \mu_j y_{rj}^u\right) \ge \left(y_{rj}^l,y_{rj}^m,y_{rj}^u\right), \quad p=1,\dots,q,\\
&\left(\sum_{i=1}^m c_i\alpha_{io}^l, \sum_{i=1}^m c_i\alpha_{io}^m, \sum_{i=1}^m c_i\alpha_{io}^u\right) = \frac{\left(\sum_{i=1}^m c_i\bar{x}_{io}^l,\ \sum_{i=1}^m c_i\bar{x}_{io}^m,\ \sum_{i=1}^m c_i\bar{x}_{io}^u\right)}{\left(CE^l,CE^m,CE^u\right)}, \ i=1,\dots,m,\\
&\lambda_j \ge 0,\ \mu_j \ge 0, \quad j=1,\dots,n.
\end{aligned} \tag{29}$$

Based on the fuzzy arithmetic approach, the optimal values at the three bounds can be calculated using the following models.

$$\begin{aligned}
&\min \ \left(\alpha_1^l,\alpha_2^l,\dots,\alpha_m^l\right)\\
&\text{s.t.} \sum_{j=1}^n \lambda_j x_{ij}^u \le \theta_{input}^u \alpha_{io}^l, i=1,\dots,m,\\
&\sum_{j=1}^n \lambda_j z_{pj}^u \ge z_{po}^u, \quad p=1,\dots,q,\\
&\sum_{j=1}^n \mu_j z_{pj}^u \le z_{po}^u, \quad p=1,\dots,q,\\
&\sum_{j=1}^n \mu_j y_{rj}^u \ge \beta_{ro}^u, \quad r=1,\dots,\ s,\\
&CE^{u^*} = \frac{\sum_{i=1}^m c_i\bar{x}_i^{u^*}}{\sum_{i=1}^m c_i\alpha_i^l}\\
&\lambda_j \ge 0, \mu_j \ge 0, \quad j=1,\dots,\ n.
\end{aligned} \tag{30}$$

$$\begin{aligned}
&\min \left(\alpha_{1o}^m,\alpha_{2o}^m,\dots,\alpha_{mo}^m\right)\\
&\text{s.t. } \sum_{j=1}^n \lambda_j x_{ij}^m \le \theta_{input}^m \alpha_{io}^m, \quad i=1,\dots,m,\\
&\sum_{j=1}^n \lambda_j z_{pj}^m \ge z_{po}^m,\\
&\sum_{j=1}^n \mu_j z_{pj}^m \le z_{po}^m,\\
&\sum_{j=1}^n \mu_j y_{rj}^m \ge \beta_{ro}^m,\\
&CE^{m^*} = \frac{\sum_{i=1}^m c_i\bar{x}_i^m}{\sum_{i=1}^m c_i\alpha_i^m}, \quad i=1,\dots,m,\\
&\lambda_j \ge 0,\ \mu_j \ge 0, \quad j=1,\dots,n.
\end{aligned} \tag{31}$$

$$\begin{aligned}
&\min \left(\alpha_{1o}^u,\alpha_{2o}^u,\dots,\alpha_{mo}^u\right)\\
&\text{s.t. } \sum_{j=1}^n \lambda_j x_{ij}^l \le \theta_{input}^l \alpha_{io}^u, \quad i=1,\dots,m,\\
&\sum_{j=1}^n \lambda_j z_{pj}^l \ge z_{po}^l, \quad p=1,\dots,q,\\
&\sum_{j=1}^n \mu_j z_{pj}^l \le z_{po}^l, \quad p=1,\dots,q,
\end{aligned} \tag{32}$$

$$\sum_{j=1}^{n}\mu_j y_{rj}^{l} \geq \beta_{ro}^{l}, \qquad r=1,\ldots,s,$$

$$CE^{l^*} = \frac{\sum_{i=1}^{m} c_i \bar{x}_i^{l*}}{\sum_{i=1}^{m} c_i \alpha_i^{u}}, \qquad i=1,\ldots,m, \tag{32}$$

$$\lambda_j \geq 0,\ \mu_j \geq 0, \qquad j=1,\ldots,n.$$

The above multi-objective model can be solved using various methods. As indicated in *Eq. (11)*, in this study, the weighted sum method is employed.

Proposed algorithm

Step 1. First, for the DMUs under consideration (DMU_O), the technical efficiency is calculated at the lower, middle, and upper bounds using the fuzzy arithmetic approach with the aid of *Models (14)-(16).*

Step 2. Assume that $c_i \in \mathbb{R}, (i = 1, \ldots, m)$ represents the cost (weight) corresponding to the inputs $x_i, (i = 1, \ldots, m)$. Then, *Models (21), (23)*, and *(25)* are solved, and using *Relations (22), (24)*, and *(26)*, the cost efficiency scores are also computed at the lower, middle, and upper bounds.

Step 3. It is assumed that the outputs of the selected workshop (DMU_O) change from $\tilde{y}_{ro}, (r = 1, \ldots, s)$ to $(\tilde{y}_{ro} + \widetilde{\Delta y}_{ro}) = \tilde{\beta}_{ro}$. *Model (27)* is then solved.

Step 4. Using *Models (30)-(32)*, the required levels of inputs are estimated at the lower, middle, and upper bounds. In this process, the overall efficiency of the system from the relational perspective, as well as the overall cost efficiency, remain unchanged, while the output levels are allowed to vary.

7 | Numerical Example

Among the industries that have a significant impact on the economy of Iran and indeed the world the oil and petrochemical industry plays a crucial role. The available oil and natural gas, through various reactions involving hydrocarbons in refineries, are converted into basic, intermediate, and final products that are used to meet a wide range of needs, such as healthcare, food, pharmaceuticals, and others. Stud bolts are of particular importance in the petrochemical industry. These bolts are vital components for ensuring the safety and stability of connections in the equipping, maintenance, and commissioning of oil and gas refineries. They are used to connect pipes, flanges, and vessels in oil and gas processing units for the transfer of fluids and chemical substances; therefore, they must have high resistance to high pressure and temperature as well as to chemical materials. It should be noted that even the slightest negligence in the stages of production, processing, or installation can lead to irreparable human and financial losses. The manufacturing workshop carefully procures raw materials in accordance with international standards, and industrial products are produced using modern technologies. These products are then tested in reputable laboratories by specialists through various tests. Subsequently, the final products, with appropriate and safe packaging, are sent to refinery equipment and commissioning workshops especially in the southern regions of the country where they are utilized by specialized engineers and technicians. In this study, a stud bolt manufacturing workshop and their application in equipping part of a refinery unit are examined.

Sub-process 1

I. Input 1: MO40 steel bar, size 16 (number of bars)

II. Input 2: MO40 steel bar, size 20 (number of bars)

III. Input 3: labor (number of workers)

IV. Output 1 (intermediate measures): stud bolt, material: ASTM A193

V. Output 2 (intermediate measures): stud bolt, material: ASTM A320

Costs

I. Cost of purchasing each MO40 steel bar, size 16: 1,400,000 tomans

II. Cost of purchasing each MO40 steel bar, size 20: 1,600,000 tomans

III. Average wage cost per worker in this project: 7,000,000 tomans

Sub-process 2

I. Input 1 (intermediate measures): stud bolt, material: ASTM A193

II. Input 2 (intermediate measures): stud bolt, material: ASTM A320

III. Final output: equipping and connecting fluid transmission pipelines (in meters): length of connected pipeline.

Table 1. First subprocess, data set.

Row	First Input of Rebar mo40 Size 16 (Number of Branches)	Second Input Mo40 Bar Size 20 (Number of Branches)	Third Entry Human Resources (Number)	First Output Intermediate Measures Stud Bolt ASTM A193 (Quantity)	Second Output Intermediate Actions Master Bolt ASEM A320 (Number)
1	(46, 45, 42)	(91, 80, 78)	(13, 9, 8)	(2800, 2500, 2100)	(5100, 4700, 4520)
2	(58, 50, 47)	(80, 77, 60)	(14, 10, 9)	(3500, 3000, 2800)	(4600, 4010, 3750)
3	(55, 52, 49)	(75, 68, 63)	(15, 10, 8)	(3050, 3010, 2720)	(4100, 3900, 3500)
4	(53, 48, 46)	(79, 75, 71)	(12, 10, 8)	(3010, 3000, 2640)	(4420, 4100, 3900)
5	(59, 55, 52)	(85, 80, 78)	(15, 12, 9)	(3500, 3100, 2920)	(4710, 4600, 3980)
6	(47, 40, 30)	(91, 82, 80)	(13, 11, 8)	(2600, 2410, 2100)	(4900, 4780, 4100)
7	(46, 45, 38)	(83, 81, 70)	(6, 5, 4)	(3540, 3100, 2910)	(5800, 5100, 4910)
8	(65, 52, 50)	(90, 82, 81)	(15, 14, 13)	(2920, 2800, 2120)	(3100, 3000, 2800)
9	(41, 39, 37)	(73, 70, 65)	(5, 4, 3)	(2600, 2460, 2340)	(4500, 4200, 3840)
10	(50, 40, 35)	(90, 79, 77)	(15, 10, 9)	(2900, 2500, 2090)	(5150, 4720, 4600)
11	(49, 41, 37)	(75, 68, 63)	(14, 10, 9)	(2800, 2200, 2100)	(4420, 4100, 3900)
12	(55, 53, 49)	(79, 75, 71)	(15, 10, 8)	(3100, 3020, 2710)	(4710, 4600, 3980)
13	(33, 32, 30)	(85, 80, 78)	(12, 10, 8)	(3500, 3100, 2920)	(4900, 4780, 4100)
14	(54, 51, 46)	(91, 82, 80)	(15, 12, 9)	(2600, 2410, 2100)	(5800, 5100, 4910)
15	(59, 54, 53)	(83, 81, 70)	(13, 11, 8)	(3540, 3100, 2910)	(4100, 3900, 3500)
16	(48, 41, 31)	(91, 82, 80)	(12, 10, 8)	(2920, 2800, 2120)	(4420, 4100, 3900)
17	(58, 51, 48)	(83, 81, 70)	(15, 12, 9)	(3010, 3000, 2640)	(4710, 4600, 3980)
18	(42, 40, 37)	(90, 82, 81)	(13, 11, 8)	(3500, 3100, 2920)	(4900, 4780, 4100)
19	(59, 53, 46)	(73, 70, 65)	(14, 10, 9)	(2600, 2410, 2100)	(5800, 5100, 4910)
20	(40, 39, 38)	(90, 79, 77)	(15, 10, 8)	(3540, 3100, 2910)	(3100, 3000, 2800)

Table 2. First subprocess, data set.

Row	First Entry (Intermediate Measures) Master Bolt ASTM A193 (Quantity)	Second Input (Intermediate Actions): Master Bolt Screw Master Bolt ASEM320 (Number)	Final Output: Equipped Connections (in Meters)
1	(2800, 2500, 2100)	(5100, 4700, 4520)	(470, 430, 410)
2	(3500, 3000, 2800)	(4600, 4010, 3750)	(520, 483, 480)
3	(3050, 3010, 2720)	(4100, 3900, 3500)	(530, 510, 470)
4	(3010, 3000, 2640)	(4420, 4100, 3900)	(450, 400, 380)
5	(3500, 3100, 2920)	(4710, 4600, 3980)	(540, 520, 480)
6	(2600, 2410, 2100)	(4900, 4780, 4100)	(350, 320, 310)
7	(3540, 3100, 2910)	(5800, 5100, 4910)	(640, 600, 580)
8	(2920, 2800, 2120)	(3100, 3000, 2800)	(380, 370, 350)
9	(2600, 2460, 2340)	(4500, 4200, 3840)	(620, 610, 600)
10	(2900, 2500, 2100)	(5150, 4720, 4600)	(460, 420, 400)
11	(2800, 2200, 2100)	(5100, 4630, 4610)	(530, 510, 470)
12	(3100, 3020, 2710)	(4150, 3950, 3600)	(450, 400, 380)
13	(3050, 3010, 2720)	(4600, 4310, 3810)	(540, 520, 480)

Table 2. Continued.

Row	First Entry (Intermediate Measures) Master Bolt ASTM A193 (Quantity)	Second Input (Intermediate Actions): Master Bolt Screw Master Bolt ASEM320 (Number)	Final Output: Equipped Connections (in Meters)
14	(3010, 3000, 2640)	(4710, 4600, 3980)	(350, 320, 310)
15	(3500, 3100, 2920)	(4900, 4780, 4100)	(640, 600, 580)
16	(2600, 2410, 2100)	(5800, 5100, 4910)	(640, 600, 580)
17	(3540, 3100, 2910)	(3100, 3000, 2800)	(380, 370, 350)
18	(2920, 2800, 2120)	(4500, 4200, 3840)	(620, 610, 600)
19	(3540, 3100, 2910)	(3100, 3000, 2800)	(460, 420, 400)
20	(2920, 2800, 2120)	(4500, 4200, 3840)	(450, 400, 380)

Table 3. Technical efficiency table.

DMU	θ^{*l}_{input}	θ^{*m}_{input}	θ^{*u}_{input}
1	0.32	0.34	0.33
2	0.34	0.86	0.72
3	0.24	0.46	0.34
4	0.55	0.50	0.86
5	0.23	0.43	0.46
6	0.61	0.49	0.50
7	0.30	0.33	0.43
8	0.22	0.61	0.49
9	0.40	0.30	0.33
10	0.76	0.22	0.72
11	0.40	0.40	0.33
12	0.34	0.76	0.61
13	0.23	0.40	0.30
14	0.61	0.34	0.22
15	0.30	0.61	0.40
16	0.22	0.30	0.76
17	0.40	0.22	0.40
18	0.76	0.40	0.50
19	0.40	0.76	0.43
20	0.34	0.40	1

Table 4. Cost-effectiveness table.

DMU	CE^{*l}	CE^{*m}	CE^{*u}
1	0.32	0.46	0.51
2	0.35	0.50	0.64
3	0.36	0.43	0.63
4	0.32	0.49	0.49
5	0.34	0.33	0.55
6	0.24	0.72	0.41
7	0.55	0.34	0.90
8	0.23	0.86	0.36
9	0.61	0.46	1
10	0.30	0.50	0.51
11	0.22	0.43	0.37
12	0.40	0.49	1
13	0.76	0.33	0.66
14	0.40	0.72	1
15	0.34	1	0.63
16	0.61	0.50	0.55
17	0.30	0.43	0.41
18	0.22	0.49	0.90
19	0.40	0.33	0.36
20	0.60	0.84	1

In *Table 3*, the technical efficiency scores of all DMUs (workshops) have been calculated under the lower, middle, and upper bounds using the proposed models. As shown in *Table 3*, the upper-bound technical efficiency score of DMUs No. 20 is equal to one. According to *Definition 6*, this unit can therefore be classified as weakly efficient, while the remaining DMUs are inefficient.

Furthermore, suppose that the input cost vector is given as $(c_1, c_2, c_3) = (7, 1.6, 1.4)$. The cost efficiency scores have been computed using *Models (21)-(26)* for the lower, middle, and upper bounds. The results indicate that DMUs No. 9 and No. 20 achieve a cost efficiency score of one at the upper bound and are thus cost-efficient at this bound. In addition, DMUs No. 13 attains a cost efficiency score of one at both the middle and upper bounds, which may be considered an indication of the cost efficiency of this unit.

Suppose that the manager decides to change the outputs of the DMUs (workshops). Managers or engineers may wish to increase the level of outputs based on their preferred objectives. This raises the following question: to what extent should the input levels $\tilde{\alpha} = (\tilde{x}_i + \widetilde{\Delta x}_i),\ i = 1, \ldots, m$ be adjusted so that both the overall technical efficiency $\tilde{\theta}_0^*$ of the process and the cost efficiency $\widetilde{CE}_0^*$ remain constant, while the output levels are changed to the desired amounts?

The technical efficiency of all DMUs (workshops) at the lower, middle, and upper bounds has been kept constant according to *Table 5*. Meanwhile, the output levels of the workshops have been modified based on the preferences of managers and engineers, as reported in *Table 5*.

Table 5. Change in outputs in the manager's chosen DMUs.

DMU	$(\theta^{l^*}, \theta^{m^*}, \theta^{u})$	$(CE^{l^*}, CE^{m^8}, CE^{u^*})$	$(\beta^{l}, \beta^{m}, \beta^{u})$
DMU7	(0.76, 0.57, 0.48)	(0.90, 0.72, 0.55)	(590,720,900)
DMU9	(0.98, 0.71, 0.56)	(1, 0.86, 0.61)	(620,730,900)
DMU13	(0.86, 0.75, 0.71)	(1, 1, 0.76)	(710,850,950)
DMU19	(0.57, 0.45, 0.39)	(0.63, 0.49, 0.35)	(480,504,800)
DMU20	(1.00, 0.67, 0.57)	(1, 0.84, 0.60)	(612,800,900)

Table 6. Estimation of inputs with the proposed model.

DMU	$(\alpha_1^l, \alpha_2^l, , \alpha_3^l)$	$(\alpha_1^m, \alpha_2^m, , \alpha_3^m)$	$(\alpha_1^u, \alpha_2^u, , \alpha_3^u)$
DMU7	(4,75,41)	(45,97,8)	(62,114,10)
DMU9	(5,65,37)	(39,80,7)	(55,103,9)
DMU13	(40,73,5)	(41,82,6)	(50,92,7)
DMU19	(46,77,9)	(53,80,11)	(63,116,19)
DMU20	(38,65,5)	(41,92,8)	(54,99,10)

The results presented in *Table 6* indicate that if the output levels for example, for DMU 7 (Workshop 7) are increased to the values (590, 720, 900), the required input levels for the lower, middle, and upper bounds of the first, second, and third inputs will be (4, 75, 41), (45, 97, 8), and (62, 114, 10), respectively. These findings enable managers to make more informed decisions regarding the procurement of raw materials and labor, particularly in critical industries such as petrochemicals. Moreover, this approach provides a basis for formulating more realistic plans to achieve production self-sufficiency. By utilizing this information, the stages of the production process can be organized prior to implementation; as a result, quality is improved, waste is reduced, and more desirable outcomes are achieved.

8 | Conclusion

Since, in reality, public and private organizations such as hospitals, schools, banks, and similar institutions operate through two- or multi-stage processes, NDEA models should be employed to evaluate their efficiency. Classical DEA models treat the system as a black box and do not seek to identify sources of inefficiency within sub-processes. In recent decades, new network DEA models have been developed that explicitly incorporate intermediate measures. On the other hand, in empirical studies, data are not always precise and deterministic; they may be vague (fuzzy) and expressed in the form of triangular fuzzy numbers with lower, middle, and upper bounds. Consequently, network DEA models with fuzzy data have also been developed, and various solution approaches have been proposed.

In this study, a NDEA model for technical efficiency with triangular fuzzy data is proposed and solved using the fuzzy arithmetic approach. Under this approach, three linear programming problems are solved straightforwardly, yielding the lower, middle, and upper bounds of the technical efficiency of the DMUs under evaluation. Based on this framework, a cost efficiency model is also proposed, and the optimal values of cost efficiency are computed for the three bounds as well. Subsequently, an IDEA model with triangular fuzzy representations for inputs and outputs is introduced, such that overall technical efficiency and cost efficiency are preserved, while the output vector is allowed to change (be perturbed), and the required changes in inputs are estimated using the proposed model. This model serves as a powerful tool that helps managers not only evaluate past performance but also make informed and well-justified decisions with regard to future planning.

Conflict of Interest

The authors declare no conflict of interest.

Data Availability

All data are included in the text.

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