Annals of Optimization With Applications

www.anowa.reapress.com

Ann. Optim. Appl. Vol. 2, No. 1 (2026) 14–32.

Paper Type: Original Article

A Hybrid Fast Numerical Method for the Lane-Emden Differential Equation Using GHFs

Seyed Amjad Samareh Hashemia[1,*], Rasoul Hatamian[1]

[1] Department of Basic Sciences, School of Mathematical Sciences, P.O. Box 19395-3697, Payame Noor University (PNU), Tehran, Iran; a_hashemi@pnu.ac.ir.

Citation:

Received: 17 February 2024 Revised: 22 April 2024 Accepted: 24 June 2024	Samareh Hashemia, S. A., & Hatamiana, R. (2026). A hybrid fast numerical method for the lane-emden differential equation using GHFs. *Annals of optimization with applications, 2*(1), 14-32.

Abstract

This paper introduces a novel hybrid numerical method which solves the Lane-Emden equation, leveraging Generalized Hat Functions (GHFs) of degrees 1 and 3 to achieve exceptional computational efficiency. By using linear GHFs for converting the equation into a block-structured nonlinear system solved via forward substitution, followed by cubic GHFs for refined approximation, the approach delivers up to 1000x speedup over direct cubic methods while maintaining L_∞ errors around 10−4. The proposed method adaptable to various nonlinear differential equations, it ensures consistent accuracy across interval lengths and extends seamlessly to fractional-order cases with minimal adjustments.

Keywords: Generalized hat functions, Lane-emden equation, Operational matrix of integration, Numerical differential equations.

1|Introduction

The Lane-Emden equation is a fundamental tool in astrophysics used to understand and model the structure and evolution of stars. It describes the balance between gravity and the pressure gradient within a star, providing insights into key properties like temperature, density, and pressure distribution [12, 7]. This equation emerges from several basic principles [7]:

I. Hydrostatic equilibrium: Stars are massive objects, but their internal forces balance each other, preventing them from collapsing under their own gravity.

Corresponding Author: Mahdi.hamishebahar77@sharif.edu

https://doi.org/10.48314/anowa.v2i1.65

II. Newton's law of gravitation: Every particle within a star exerts gravitational attraction on others.

III. Ideal gas law: The pressure within a star is related to its temperature and density.

By combining these principles, mathematicians Jonathan Homer Lane and Robert Emden derived the Lane-Emden equation, expressing the relationship between density ρ, radius r, and a polytropic index n representing the pressure-density relation within the star. The Lane-Emden equation has numerous applications in understanding stars: [7]

I. Modeling stellar structure: By solving the equation with different boundary conditions and polytropic indices, scientists can create mathematical models for various types of stars, like main-sequence stars, white dwarfs, and neutron stars.

II. Studying stellar evolution: Understanding how the density and pressure vary within a star helps predict its behavior and evolution over time, including stages like burning fuel, collapsing, or exploding.

III. Interpreting observations: Comparing theoretical models based on the Lane-Emden equation with observations of stars (e.g., luminosity, spectra) helps astronomers infer their internal properties and processes.

While powerful, the Lane-Emden equation has limitations: [7]

I. Simplified assumptions: It assumes spherical symmetry and constant temperature within specific regions, which isn't always true for real stars.

II. Numerical solutions: Analytical solutions often exist only for specific cases, requiring numerical methods for other scenarios.

The Lane-Emden equation, despite its limitations, remains a cornerstone for understanding stellar structure and a valuable tool for astrophysicists to explore the complexities of stars. Its applications range from modeling basic star types to unraveling the mysteries of stellar evolution and interpreting observational data. By studying this equation, scientists gain deeper insights into the fascinating world of these celestial giants. The equation is given by:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n,$$

where ξ is a dimensionless radius, θ is a dimensionless function related to the density, and n is the polytropic index that determines the equation of state of the fluid. The equation has exact solutions for some values of n, such as $n = 0, 1, 5$, and numerical solutions for others [5, 11, 1, 10, 4]. The solutions, known as polytropes, can be used to approximate the density, pressure, temperature, and mass profiles of stars with different compositions and evolutionary stages. For example, a polytrope with $n = 3$ can model a white dwarf, while a polytrope with $n = 1.5$ can model a main-sequence star. The Lane-Emden equation is important because it provides a simple way to understand the basic properties and stability of stars, without solving the full set of equations of stellar structure. It also reveals some universal features of self-gravitating systems, such as the mass-radius relation and the critical mass for collapse. The equation can also be applied to other astrophysical phenomena, such as planetary atmospheres, neutron stars, dark matter halos, and galaxy clusters.

A wide range of numerical techniques have been applied to this eqution. An extensive array of numerical techniques has been devised and implemented for the solution of this equation. Notable examples of these include the following papers: The Taylor wavelet method is utilized to solve the equation as described in [4]. Studies [1], [6] and [8] propose numerical methods for Lane-Emden equations. These methods utilize a collocation approach based on various types of B-splines, coupled with the quasilinearization technique to handle the nonlinearities. A novel machine learning approach for the solution of Lane-Emden type differential equations is introduced in [5], employing Rational Chebyshev polynomials. The authors develop and improved a higher-order compact finite difference scheme specifically designed to solve systems of equations of the Lane-Emden-Fowler type [11, 3]. The authors proposed an Artificial Neural Network (ANN) approach for the solution of the Equation in [10].

This paper presents a novel numerical approach for solving the Lane-Emden equation, broadening its applicability to a diverse array of nonlinear differential equations. The proposed method leverages Generalized Hat functions in combination with their operational matrix of integration to transform the Lane-Emden equation into a structured nonlinear system of equations. This resulting system is then effectively solved through forward substitution. To further enhance the precision of the solution, quintic Hermite interpolation is applied, yielding an accurate approximation of the equation's solution.

One of the standout features of this method is its exceptional flexibility. Unlike many traditional techniques, the accuracy of the proposed approach remains consistent regardless of the length of the solution interval, making it highly robust. Additionally, with minor adjustments, the method can be extended to address fractional-order Lane-Emden equations, showcasing its adaptability. Beyond Lane-Emden equations, the approach is versatile enough to handle a broad spectrum of nonlinear ordinary differential equations.

The method provides multiple pathways for deriving approximate solutions, with Hermite interpolation proving particularly effective in achieving superior accuracy. The paper offers a thorough analysis of these methods, supported by detailed error assessments, to highlight their comparative strengths and weaknesses. To validate the proposed approach, numerical examples are included, demonstrating its accuracy, efficiency, and adaptability across various problem settings. This innovative method opens new avenues for tackling complex differential equations with improved precision and computational efficiency.

2|Preliminary Concepts

In this section, we introduce the necessary mathematical tools for implementing the proposed method. These consist of the degree-1 and degree-3 generalized hat functions for function approximation, along with their essential properties and the associated integral operational matrices.

2.1|Generalized Hat Functions (GHFs)

In this section we recall the generalized hat functions and their properties. Generalaized Hat functions (GHFs) are continuous functions with shape of hats (in case of degree one). Suppose that the interval $[0, T]$, for $T > 0$, is divided into N subintervals $[ih, (i+1)h], \; i = 0, 1, \cdots, N-1$ of equal lengths h where $h = \frac{T}{N}$. For more details on GHFs and their propertied refer to [2] and its references.

The GHFs from first through forth degrees are defined as the following, respectively:

First degree (Linear) GHFs definition

First-degree GHFs were the building blocks upon which the broader concept of GHFs was constructed. The GHFs of degree one are defined as: [2]

$$\psi_0(t) = \begin{cases} \frac{h-t}{h}, & 0 \le t < h, \\ 0, & \text{otherwise.} \end{cases}$$

$$\psi_i(t) = \begin{cases} \frac{t-(i-1)h}{h}, & (i-1)h \le t < ih, \\ \frac{(i+1)h-t}{h}, & ih \le t < (i+1)h, \quad i = 1, 2, \cdots, N-1 \\ 0, & \text{otherwise.} \end{cases}$$

$$\psi_N(t) = \begin{cases} \frac{t-(T-h)}{h}, & T-h \le t \le T, \\ 0, & \text{otherwise.} \end{cases}$$

Third degree (Cubic) GHFs definition

[2] Suppose N is a positive integer of multiple three ($N = 3m,\ m \in \mathbb{N}$) and $h = \frac{T}{N}$. A set of adjust GHFs of degree 3 is defined on $[0, T]$ as:

$$\psi_0(t) = \begin{cases} \frac{-1}{6h^3}(t-h)(t-2h)(t-3h), & 0 \le t < 3h, \\ 0, & \text{otherwise.} \end{cases}$$

If $i = 3k+1,\ k = 0, 1, \cdots, m-1$:

$$\psi_i(t) = \begin{cases} \frac{1}{2h^3}(t-(i-1)h)(t-(i+1)h)(t-(i+2)h), & (i-1)h \le t < (i+2)h, \\ 0, & \text{otherwise.} \end{cases}$$

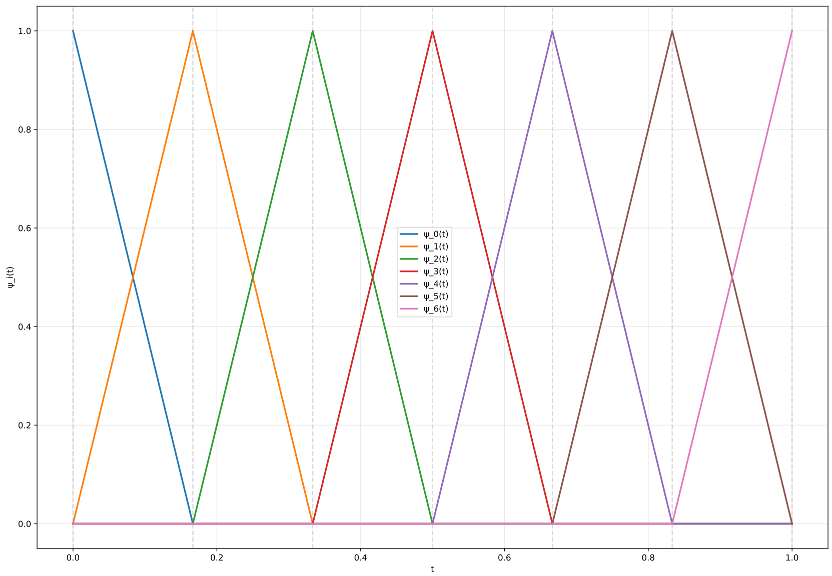
If $i = 3k+2,\ k = 0, 1, \cdots, m-1$:

$$\psi_i(t) = \begin{cases} \frac{-1}{2h^3}(t-(i-2)h)(t-(i-1)h)(t-(i+1)h), & (i-2)h \le t < (i+1)h, \\ 0, & \text{otherwise.} \end{cases}$$

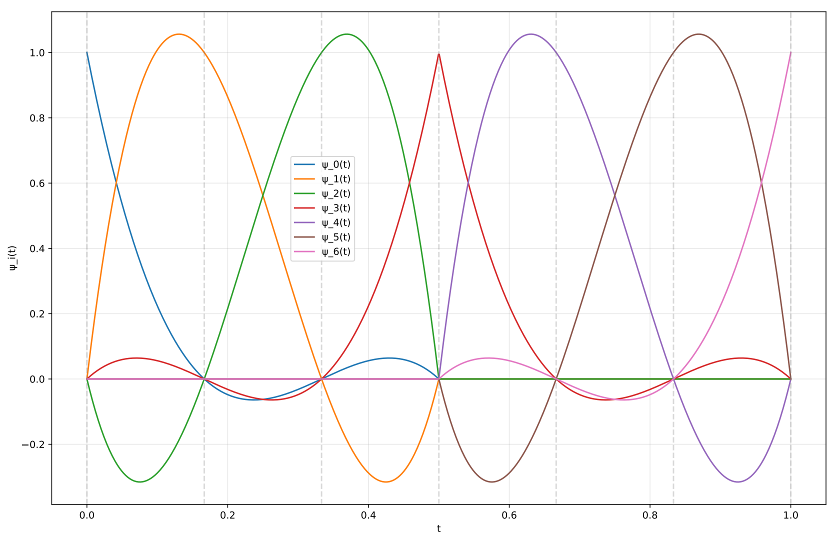
If $i = 3k,\ k = 0, 1, \cdots, m-1$:

$$\psi_i(t) = \begin{cases} \frac{1}{6h^3}(t-(i-3)h)(t-(i-2)h)(t-(i-1)h), & (i-3)h \le t < ih, \\ \frac{-1}{6h^3}(t-(i+1)h)(t-(i+2)h)(t-(i+3)h), & ih \le t < (i+3)h, \\ 0, & \text{otherwise.} \end{cases}$$

$$\psi_N(t) = \begin{cases} \frac{1}{6h^3}(t-(T-h))(t-(T-2h))(t-(T-3h)), & T-3h \le t \le T, \\ 0, & \text{otherwise.} \end{cases}$$



(a) Linear



(b) Cubic

Fig. 1. Linear and Cubic Generalized Hat Functions with $N = 6$ and $T = 1$.

2.2 | GHFs properties and their operational matrix of integration

It is easy to see that ψ_i's belong to $L^2[0,T]$ and are linearly independent, also have the following properties:

$$\psi_i(kh) = \delta_{ik}, \quad i,k = 0,1,\ldots,N, \qquad \sum_{i=0}^{N} \psi_i(t) = 1, \tag{1}$$

where δ_{ik} is the Kronecker delta.

Any function $y(t) \in L^2[0,T]$ can be approximated in terms of GHFs as:

$$y(t) \simeq y_N(t) = \sum_{k=0}^{N} c_k \psi_k(t) = \mathbf{C}_y^T \mathbf{\Psi}(t) = \mathbf{\Psi}^T(t)\mathbf{C}_y, \tag{2}$$

where, $\mathbf{\Psi}(t) = [\psi_0(t), \psi_1(t), \cdots, \psi_N(t)]^T$ and $\mathbf{C}_y = [c_0, c_1, \cdots, c_N]^T$ and the coefficients c_k's in (2) are given by:

$$c_k = y(kh), \quad k = 0, 1, \cdots, N. \tag{3}$$

According to (2) and (3), the product of the two functions can be approximated in terms of GHFs as follows:

$$f(t)g(t) \simeq \sum_{k=0}^{N} f(t_k)g(t_k)\psi_k(t) = (C_f \odot C_g)^T \mathbf{\Psi}(t), \tag{4}$$

where $\odot$ is Hadamard product (elementwise product), therefore, for $r \geq 0$,

$$(f(t))^r \simeq \sum_{k=0}^{N} (f(t_k))^r \psi_k(t) = ((C_f)^r_{\odot})^T \mathbf{\Psi}(t). \tag{5}$$

Using GHFs, one can approximate integral of GHFs, hence

$$\int_0^t \mathbf{\Psi}(s)\, ds \simeq \mathbf{P}\mathbf{\Psi}(t),$$

where, $\mathbf{P}$ is a $(N+1)\times(N+1)$ matrix called operational matrix of integration that is given for various degrees of GHFs in sequel.

Operation matrix of integration for linear GHFs

[2] For $N \in \mathbb{N}$, $\mathbf{P}$ is obtained as follows:

$$\mathbf{P} = \frac{h}{2}\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 & \dots & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 & \dots & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & \dots & 2 & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

Operation matrix of integration for cubic GHFs

[2] For $N = 3m, \quad m \in \mathbb{N}$, $\mathbf{P}$ is obtained as:

$$\mathbf{P} = \frac{h}{144}\begin{bmatrix} 0 & \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_2 & \mathbf{p}_2 & \mathbf{p}_2 & \dots & \mathbf{p}_2 & \mathbf{p}_2 \\ \mathbf{0}_{3\times1} & \mathbf{D} & \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_2 & \mathbf{N}_2 & \dots & \mathbf{N}_2 & \mathbf{N}_2 \\ \mathbf{0}_{3\times1} & \mathbf{0}_{3\times3} & \mathbf{D} & \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_2 & \dots & \mathbf{N}_2 & \mathbf{N}_2 \\ \mathbf{0}_{3\times1} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{D} & \mathbf{N}_1 & \mathbf{N}_2 & \dots & \mathbf{N}_2 & \mathbf{N}_2 \\ \mathbf{0}_{3\times1} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{D} & \mathbf{N}_1 & \dots & \mathbf{N}_2 & \mathbf{N}_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_{3\times1} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \dots & \mathbf{D} & \mathbf{N}_1 \\ \mathbf{0}_{3\times1} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \dots & \mathbf{0}_{3\times3} & \mathbf{D} \end{bmatrix}_{(N+1)\times(N+1)},$$

where, $\mathbf{p}_1 = [54, 48, 54]$, $\mathbf{p}_2 = [54, 54, 54]$,

$$\mathbf{D} = \begin{bmatrix} 114 & 192 & 162 \\ -30 & 48 & 162 \\ 6 & 0 & 54 \end{bmatrix}, \quad \mathbf{N}_1 = \begin{bmatrix} 162 & 162 & 162 \\ 162 & 162 & 162 \\ 108 & 102 & 108 \end{bmatrix} \quad \text{and} \quad \mathbf{N}_2 = \begin{bmatrix} 162 & 162 & 162 \\ 162 & 162 & 162 \\ 108 & 108 & 108 \end{bmatrix}.$$

3|The Primary Method

Consider the following second order singular ordinary differential equation with initial conditions

$$y''(x) + \frac{\alpha}{t}y'(x) + y^n(x) = 0, \qquad 0 \le x \le T, \tag{6}$$

$$y(0) = \beta, \quad y'(0) = \eta, \tag{7}$$

where α, T, β and η are given numbers and n is a known nonnegative integer. This second order singular ordinary differential equation is known as the

Lane-Emden equation[9]. We will find the approximate solution using the numerical methods that we discussed earlier as well as another one. For the cases where $n = 0, 1, 5$, this equation possesses a closed-form solution while for other values of n, a closed-form solution is not known. With $\alpha = 2,\ \beta = 1,\ \eta = 0$ the exact solution of the equation is:

$$
\begin{aligned}
n &= 0, \qquad y(x) = 1 - \frac{x^2}{6},\\
n &= 1, \qquad y(x) = \frac{\sin x}{x},\\
n &= 5, \qquad y(x) = \frac{1}{\sqrt{1 + \frac{1}{3}x^2}}.
\end{aligned}
$$

Initially, by taking the limit of the equation (6) and using L'Hopital's rule, we have:

$$
\begin{aligned}
&\lim_{x\to 0}(y''(x) + \frac{\alpha}{x}y'(x) + (y(x))^n) = 0\\
&\Rightarrow \quad y''(0) + \alpha y''(0) + (y(0))^n = 0\\
&\Rightarrow \quad y''(0) = \frac{-\beta^n}{1+\alpha}. \qquad (8)
\end{aligned}
$$

Then, we rewrite the equation (6) as $xy'' + \alpha y' + xy^n = 0$ and consider an approximate solution for $y''(x)$ based on the given basis, i.e. GHFs, as follows:

$$
y''(x) \simeq \sum_{i=0}^{N} c_i \psi(x) = C_{y''}^T \mathbf{\Psi}(x), \qquad (9)
$$

therefore, we will have the followings facts about $y'(x)$ and $y(x)$,

$$
\begin{aligned}
y'(x) &= \int_0^x y''(t)\,dt + y'(0) \qquad (10)\\
&\simeq C_{y''}^T \mathbf{P}\mathbf{\Psi}(x) + \eta, \qquad (11)\\
y(x) &= \int_0^x y'(t)\,dt + y(0) \qquad (12)\\
&\simeq C_{y''}^T \mathbf{P}^2\mathbf{\Psi}(x) + \eta x + \beta, \qquad (13)
\end{aligned}
$$

where $\mathbf{P}$ is the operational matrix of integral of the selected GHFs. Besides, we expand the functions $f(x) = 1$ and $g(x) = x$ in terms of GHFs. So, we will have $1 = J^T\mathbf{\Psi}(x)$ and and $x = E^T\mathbf{\Psi}(x)$ where $J = [1, 1, \ldots, 1]^T$ an $N+1$-vector and $E = [x_0, x_1, \ldots, x_N]^T$ where $x_i = ih,\ i = 0, 1, \ldots, N$ are the knots of GHFs. By substituting them in (11) and (13) we have:

$$y'(x) \simeq (C_{y''}^T\mathbf{P} + \eta J^T)\mathbf{\Psi}(x) = C_{y'}^T\mathbf{\Psi}(x), \tag{14}$$

$$y(x) \simeq (C_{y''}^T\mathbf{P}^2 + \eta E^T + \beta J^T)\mathbf{\Psi}(x) = C_y^T\mathbf{\Psi}(x), \tag{15}$$

where $C_{y'} = \mathbf{P}^T C_{y''} + \eta J$ and $C_y = (\mathbf{P}^T)^2 C_{y''} + \eta E + \beta J$ then substituting in (6) using (5) we will have:

$$(E \odot C_{y''})^T\mathbf{\Psi}(x) + \alpha(C_{y''}^T\mathbf{P} + \eta J^T)\mathbf{\Psi}(x) + E^T \odot (C_{y''}^T\mathbf{P}^2 + \eta E^T + \beta J^T)_\odot^n\mathbf{\Psi}(x) = 0,$$

or equivalently,

$$(E \odot C_{y''}) + \alpha(\mathbf{P}^T C_{y''} + \eta J) + E \odot ((\mathbf{P}^T)^2 C_{y''} + \eta E + \beta J)_\odot^n = 0. \tag{16}$$

System resulting from the equation (16) is a nonlinear block system. It is a nonlinear system of multi-variable polynomial equations, and its solution can be determined by employing a suitable numerical method.

The first equation of the system (16) is given by $0 \cdot c_0 = 0$, from which the value of c_0 cannot be determined. To evaluate c_0, we use the relation (3), which implies that $c_0 = y''(0)$. Thus, by employing equation (8), we obtain

$$c_0 = y''(0) = \frac{-\beta^n}{1+\alpha}, \tag{17}$$

where c_0, β, and α are defined in (9) and (7), respectively.

The number of equations within each block corresponds to the degree of the selected Generalized Hat Functions (GHFs). Furthermore, each equation within a block is a multi-variable polynomial whose degree corresponds to that of the specific Lane-Emden equation under consideration, n. The number of unknowns in the first block equals the degree of the selected GHFs. Subsequently, the number of unknowns in each successive block increases by the degree of the GHFs compared to the preceding block. This means that in the first block, the number of equations and the number of unknowns are the same, and the block equations can be solved independently from the others. Hence, the system can be solved using forward substitution. For instance, with $\alpha = 2,\ \beta = 1,\ \eta = 0,\ n = 5,\ N = 12$ and using cubic GHFs, resulting system, after simplifications, is as follows:

$$\begin{aligned}
&\frac{1}{2^{18}\cdot 3^{24}}(171c_1 - 81c_2 + 21c_3 + 3853)^5 + 31c_1 - 5c_2 + c_3 = 3,\\
&\frac{1}{2^{5}\cdot 3^{23}}(81c_1 - 9c_2 + 3c_3 + 475)^5 + 12c_1 + 12c_2 = 1,\\
&\frac{1}{2^{18}\cdot 3^{9}}(45c_1 + 9c_2 + 3c_3 + 139)^5 + 9c_1 + 9c_2 + 15c_3 = 1,
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\frac{1}{2^{16}\cdot 3^{19}}(567c_1 + 243c_2 + 116c_3 + 57c_4 - 27c_5 + 7c_6 + 1233)^5\\
&\qquad + 27c_1 + 27c_2 + 18c_3 + 67c_4 - 5c_5 + c_6 = 3,\\
&\frac{5}{2^{18}\cdot 3^{19}}(729c_1 + 405c_2 + 223c_3 + 216c_4 - 24c_5 + 8c_6 + 1215)^5\\
&\qquad + 27c_1 + 27c_2 + 17c_3 + 32c_4 + 68c_5 = 3,\\
&\frac{1}{2^{17}\cdot 3^{9}}(99c_1 + 63c_2 + 36c_3 + 45c_4 + 9c_5 + 3c_6 + 133)^5\\
&\qquad + 9c_1 + 9c_2 + 6c_3 + 9c_4 + 9c_5 + 27c_6 = 1,
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\frac{7}{2^{18}\cdot 3^{19}}(1053c_1 + 729c_2 + 432c_3 + 567c_4 + 243c_5 + 116c_6 + 57c_7 - 27c_8 + 7c_9 + 1179)^5\\
&\qquad + 27c_1 + 27c_2 + 18c_3 + 27c_4 + 27c_5 + 18c_6 + 103c_7 - 5c_8 + c_9 = 3,\\
&\frac{1}{2^{15}\cdot 3^{19}}(1215c_1 + 891c_2 + 540c_3 + 729c_4 + 405c_5 + 223c_6 + 216c_7 - 24c_8 + 8c_9 + 1161)^5\\
&\qquad + 27c_1 + 27c_2 + 18c_3 + 27c_4 + 27c_5 + 17c_6 + 32c_7 + 104c_8 = 3,\\
&\frac{1}{2^{18}\cdot 3^{8}}(153c_1 + 117c_2 + 72c_3 + 99c_4 + 63c_5 + 36c_6 + 45c_7 + 9c_8 + 3c_9 + 127)^5\\
&\qquad + 9c_1 + 9c_2 + 6c_3 + 9c_4 + 9c_5 + 6c_6 + 9c_7 + 9c_8 + 39c_9 = 1,
\end{aligned} \tag{20}$$

$$\begin{aligned}
&\frac{5}{2^{17}\cdot 3^{19}}(1539c_1+1215c_2+756c_3+1053c_4+729c_5+432c_6+567c_7+243c_8\\
&\qquad+116c_9+57c_{10}-27c_{11}+7c_{12}+1125)^5+27c_1+27c_2+18c_3+27c_4\\
&\qquad+27c_5+18c_6+27c_7+27c_8+18c_9+139c_{10}-5c_{11}+c_{12}=3,\\
&\frac{11}{2^{18}\cdot 3^{19}}(1701c_1+1377c_2+864c_3+1215c_4+891c_5+540c_6+729c_7+405c_8\\
&\qquad+223c_9+216c_{10}-24c_{11}+8c_{12}+1107)^5+27c_1+27c_2+18c_3+27c_4\\
&\qquad+27c_5+18c_6+27c_7+27c_8+17c_9+32c_{10}+140c_{11}=3,\\
&\frac{1}{2^{16}\cdot 3^{9}}(207c_1+171c_2+108c_3+153c_4+117c_5+72c_6+99c_7+63c_8\\
&\qquad+36c_9+45c_{10}+9c_{11}+3c_{12}+121)^5+9c_1+9c_2+6c_3+9c_4\\
&\qquad+9c_5+6c_6+9c_7+9c_8+6c_9+9c_{10}+9c_{11}+51c_{12}=1,
\end{aligned} \tag{21}$$

As can be seen, the nonlinear system of equations possesses a block structure. Each block consists of three equations (corresponding to the selected GHF degree), and each equation is a fifth-degree polynomial (matching the degree of the Lane–Emden equation under consideration). The equations of the first block (18) involve three unknowns (equal to the chosen GHF degree). The equations of the second block (19) involve six unknowns (the unknowns from the previous block plus three new ones). Similarly, the third block (20) introduces three additional unknowns on top of those from the second block. Finally, the forth block (21) contains all 12 unknowns.

Now, as an illustration, we solve the system of equations (18)–(21) step by step. At first, solving the first block (18) yields

$$\begin{aligned}
c_1&=-0.2813862219221314,\\
c_2&=-0.16611584006682822,\\
c_3&=-0.054115181352652905,
\end{aligned}$$

Substituting this solution into the second block (19) and solving it, gives

$$\begin{aligned}
c_4&=0.019025498660954224\\
c_5&=0.054983101733303306,\\
c_6&=0.06662123620096862,
\end{aligned}$$

And, substituting these solutions into the third block (20) and solving it, gives

$c_7 = 0.06583534770547635,$

$c_8 = 0.059721149846784255,$

$c_9 = 0.0520507709028683,$

Finally, substituting the solutions from (18), (19) and (20) into the forth (last) block (21) yields

$c_{10} = 0.04449558456131838,$

$c_{11} = 0.03772544194215955,$

$c_{12} = 0.03192077075915512.$

Moreover, as derived in (17), we obtain $c_0 = -\frac{1}{3}$. Having determined $C_{y''}$, and using (14) and (15), we can subsequently construct $C_{y'}$ and C_y, respectively.

As mentioned, we are dealing with a nonlinear system of multi-variable polynomial equations with a block structure, and the first block is as follows:

$$\begin{cases} p_{1,1}(c_1, c_2, c_3) = 0, \\ p_{1,2}(c_1, c_2, c_3) = 0, \\ p_{1,3}(c_1, c_2, c_3) = 0, \end{cases}$$

where $p_{1,i}(c_1, c_2, c_3), \quad i = 1, 2, 3$ are plynomials of degree n, n is the degree of Lane-Emden equation (6).

The above system of nonlinear equations can be solved numerically using various methods such as the Newton-Raphson method, Broyden's method, Fixed-Point iteration method, Homotopy and continuation methods, the Levenberg-Marquardt algorithm, Trust-Region Methods, or others.

When solutions are determined, substitute them into the next block which has the following form which consists of k multi-variable polynomials equations each of degree n and has $6 = 2 \times 3$ variables $c_1, \ldots, c_6$,

$$\begin{cases} p_{2,1}(c_1, \ldots, c_6) = 0, \\ p_{2,2}(c_1, \ldots, c_6) = 0, \\ p_{2,3}(c_1, \ldots, c_6) = 0, \end{cases}$$

resulting in a system of 3 polynomial equations with 3 unknowns c_4, c_5, c_6. By solving this system similarly to the previous step, the values of c_4, c_5, c_6 will be determined. In the same manner, by substituting the solutions obtained from solving the previous blocks into the next block, a system with 3 equations in 3 unknowns will be obtained. By solving this system block by block, its solutions will eventually be determined.

The primary drawback of the approach described above is its computational cost, as it requires solving several nonlinear systems with multiple unknowns. While this method offers the advantage of higher accuracy, it is computationally intensive. For instance, in the example considered, even a modest problem size of $N = 12$ requires approximately 75 seconds to converge to a solution. The significant advantage of this method for solving the system (16) (which can be a large system, $N \gg k$ with nonlinear multivariable polynomial equations in N unknowns), is that the solution of a very large system of equations can be obtained by solving much smaller systems, without the need for very time-consuming and complex calculations.

4|The Modified (Fast) Method

Now, we introduce a modification to the described method that significantly accelerates its execution. For large values of N (specifically, $N > 100$), the proposed algorithm achieves a speedup factor of nearly 10,000, albeit with a marginal loss of accuracy.

As we know from (3), the coefficients in our chosen basis are the function values at the nodes - a fact that holds for any basis degree - the coefficient vector is inherently independent of the degree. We exploit this feature to simplify the nonlinear system derived from the differential equation. In the initial step, we replace the cubic basis (and its corresponding integral operational matrix) with a linear basis, this means that we use operational matrix of integration of linear GHFs in place of the corresponding one for cubic GHFs in (16). This transforms the problem from a computationally expensive system of three nonlinear equations into a simple polynomial equation, which requires far less time to solve. Once the single-variable polynomial equations obtained in the previous step are solved, we construct an approximation to the solution of the differential equation. At this stage, rather than continuing with the linear basis, we return to cubic GHFs and employ them to produce the approximate solution. This refinement yields a smoother and more accurate solution, while simultaneously reducing the computational time by a factor of approximately 10^{-4}.

In addition to the theoretical analysis of error, solved examples also support the fact that this method yields a more accurate solution than the conventional method (using linear GHFs). Besides, the method has the advantage of less computational complexity and subsequently less time-consuming relative to the method using cubic GHFs at the initial step. A test case using the Lane-Emden equation with $n = 5$ highlights the method's primary advantages:

The main advantages of the algorithm are:

I. Speed: It is exceptionally fast due to the use of linear GHFs operational matrix of integration for the system-solving step and the forward substitution approach.

II. Efficiency: The method maintains a low runtime even for large values of N, making it highly scalable.

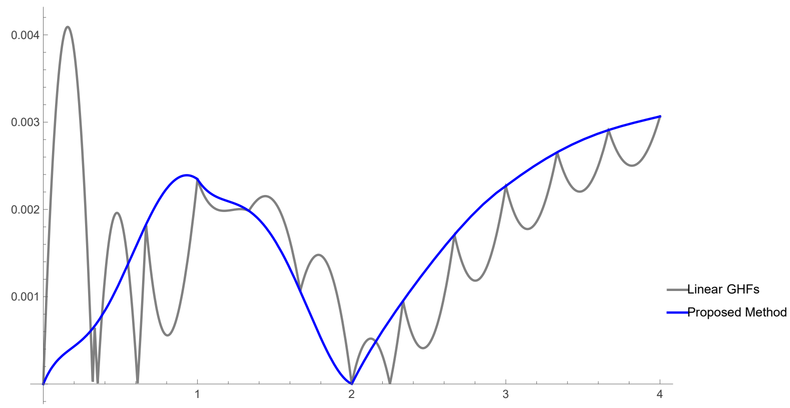
III. Adaptability: The framework can be readily extended to solve fractional-order Lane-Emden equations and other types of nonlinear differential equations with minimal changes.

IV. Simplicity: The algorithm's structure is straightforward, allowing for easy implementation in various programming environments.

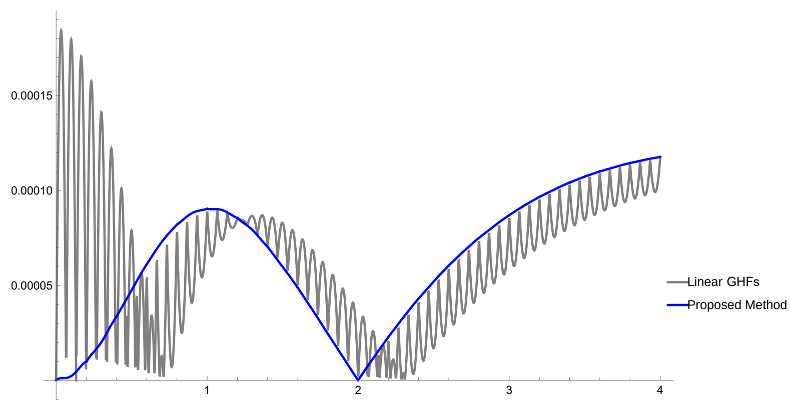
5|Illustrative Examples

At this section we present an example and solve it by both, the primary and the modified methods, with various values of N, number of knots, to compare the time and accuracy of them. The codes to solve the presented example is implemented in "WOLFRAM MATHEMATICA 14.3" in a desktop PC with the following configuration. PROCESSOR: AMD A8-9600 RADEON R7, 10 COMPUTE CORES 4C+6G 3.10 GHz, and RAM: 16 GB, 2400 MHz.

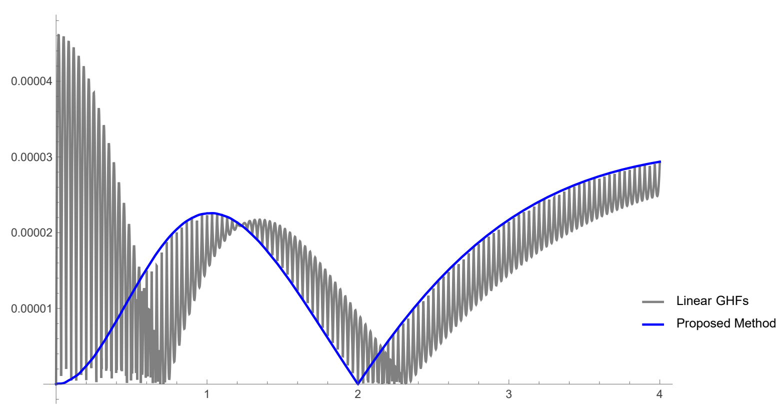
Example 1. *In this problem we consider the Lane-Emden equation* (6) *and initial conditions* (7) *with* $n = 5,\ T = 4,\ \alpha = 2,\ \beta = 0$ *and* $\eta = 1$ *whose analytical solution is* $y(x) = \frac{1}{\sqrt{1+\frac{1}{3}x^2}}$. *The proposed method and the direct method with degree 3 GHFs are applied with* $N = 12,\ 60,\ 120$ *and the results are summarized in figure 2 and table 1 to comparison.*



(a) $N = 12$

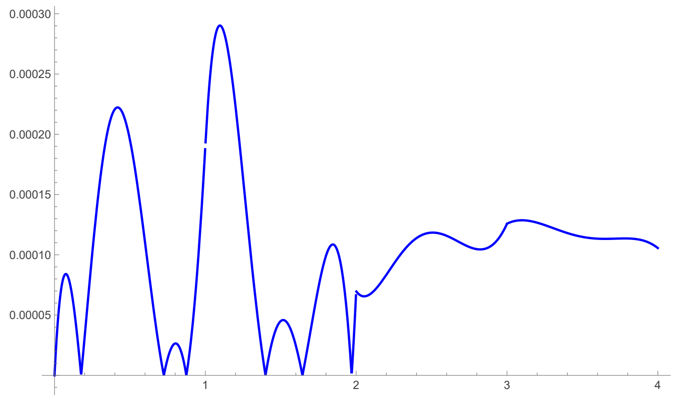


(b) $N = 60$

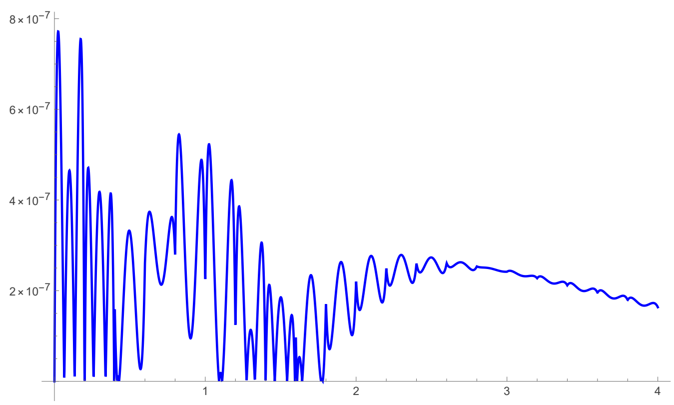


(c) $N = 120$

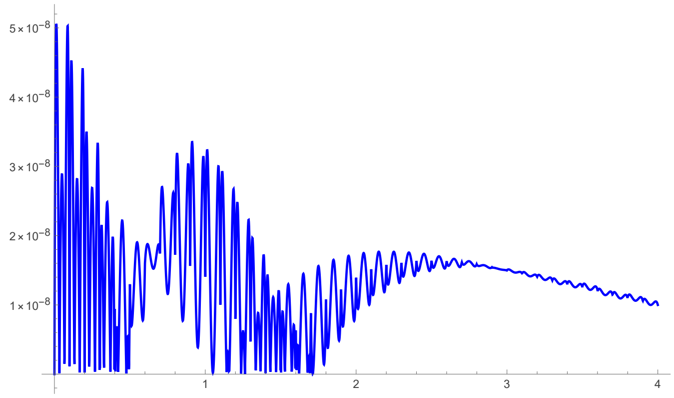
Fig. 2. Error of produced solutions using (only) linear GHFs vs. the proposed method.



(a) $N = 12$



(b) $N = 60$



(c) $N = 120$

Fig. 3. Error of produced solutions using (only) cubic GHFs.

Table 1. CPU Time and L_∞ Error comparison.

	Proposed Method	Direct Method (3rd-degree GHFs)	N
CPU time (s.) L_∞ error	0.03125 3.07×10^{-3}	88.4152 2.90×10^{-4}	12
CPU time (s.) L_∞ error	0.421875 1.17461×10^{-4}	451.719 7.665×10^{-7}	60
CPU time (s.) L_∞ error	2.09375 2.94×10^{-5}	887.438 5.0×10^{-8}	120

As shown in *Table 1*, the proposed method is orders of magnitude faster than the direct method using 3rd-degree GHFs, trading a small amount of accuracy for a significant gain in computational speed.

Acknowledgments

The authors appreciate the valuable insights and contributions of the experts who participated in this study.

Funding

This research received no external funding.

Data Availability

The data used in this study are available from the corresponding author upon reasonable request.

References

[1] Alam, M. P., Begum, T., & Khan, A. (2021). A high-order numerical algorithm for solving Lane–Emden equations with various types of boundary conditions. *Computational and applied mathematics, 40*(6), 204. https://doi.org/10.1007/s40314-021-01591-7

[2] Hashemi, A., Hatamian, R., (2025). A hybrid approach to solving the Lane-Emden differential equation using generalized hat functions and hermite interpolation, *Computational Methods for Differential Equations (CMDE)*, 1-18. https://doi.org/10.22034/cmde.2025.67759.3240

[3] Doostaki, R., Hosseini, M. M., & Salemi, A. (2024). An efficient high-order compact finite difference scheme for Lane-Emden-type equations. *Computational methods for differential equations, 13*(1), 107-122. https://doi.org/10.22034/cmde.2023.56275.2351

[4] Gümgüm, S. (2020). Taylor wavelet solution of linear and non-linear Lane-Emden equations. Applied Numerical Mathematics, 158, 44-53. https://doi.org/10.1016/j.apnum.2020.07.019

[5] He, J., Zheng, Z., & Du, C. (2023). A new constructing rational functions method for solving lane-emden type equations. *Neural processing letters, 55*(2), 1889-1918. https://doi.org/10.1007/s11063-022-10968-6

[6] Lakestani, M., & Dehghan, M. (2013). Four techniques based on the B–spline expansion and the collocation approach for the numerical solution of the Lane–Emden equation. *Mathematical methods in the applied sciences*, *36*(16), 2243-2253. https://doi.org/10.1002/mma.2755

[7] Maciel, W. J. (2015). *Introduction to stellar structure*. Springer. https://doi.org/10.1007/978-3-319-16142-6

[8] Mohammadzadeh, R., Lakestani, M., & Dehghan, M. (2014). Collocation method for the numerical solutions of Lane–Emden type equations using cubic Hermite spline functions. *Mathematical methods in the applied sciences, 37*(9), 1303-1717. https://doi.org/10.1002/mma.2890

[9] Nagle, R. K., Saff, E. B., & Snider, A. D. (2014). *Fundamentals of differential equations and boundary value problems* (pp. I-II). Pearson Education. https://www.just.edu.jo/

[10] Parand, K., Aghaei, A. A., Kiani, S., Zadeh, T. I., & Khosravi, Z. (2024). A neural network approach for solving nonlinear differential equations of Lane–Emden type. *Engineering with Computers, 40*(2), 953-969. https://doi.org/10.1007/s00366-023-01836-5

[11Sahoo, N., Singh, R., & Ramos, H. (2025). An innovative fourth-order numerical scheme with error analysis for Lane-Emden-Fowler type systems. *Numerical algorithms, 99*(1), 411-439. https://doi.org/10.1007/s11075-024-01882-0

[12] Singh, R., Singh, G., & Singh, M. (2021). Numerical algorithm for solution of the system of Emden–Fowler type equations. *International journal of applied and computational mathematics, 7*(4), 136. https://doi.org/10.1007/s40819-021-01066-7