Annals of Optimization With Applications

www.anowa.reapress.com

Ann. Optim. Appl. Vol. 2, No. 1 (2026) 33–47.

Paper Type: Original Article

Existence of Solutions for Sequential Liouville-Caputo Fractional Differential Equations

Rahmat Darzi[1,*], Bahram Agheli[2]

[1] Department of Mathematics, NeK.C., Islamic Azad University, Neka, Iran; Rahmat.Darzi@iau.ac.ir.

[2] Department of Mathematics, QaS.C., Islamic Azad University, Qaemshahr, Iran; b.agheli@qaemshahriau.ac.ir.

Citation:

Received: 17 March 2025 Revised: 18 May 2025 Accepted: 24 July 2025	Darzia, R., & Aghelib, B.(2026). Existence of solutions for sequential liouville-caputo fractional differential equations. *Annals of Optimization With Applications*, 2(1), 33-47.

Abstract

The focus of this research involves a distinct category of fractional differential equations (FDEs), specifically sequential Liouville-Caputo FDEs, which incorporate antiperiodic boundary conditions and Rie-mann-Liouville integral constraints, provided that certain appropriate conditions are satisfied. The main objective of this paper is to investigate the existence and uniqueness (EU) of the solution for the proposed problem, utilizing fixed point (FP) theory, and several novel equalities have been established in the norm form.

Keywords: Liouville-caputo derivative, Antiperiodic, integral conditions, Existence.

1 | Introduction

There seems to be a substantial growth in the field of fractional differential equations (FDE), highlighting the significance and applicability of integrals and derivatives of arbitrary order across various domains of knowledge. It is important to emphasize that fractional calculus is widely applied in natural phenomena such as chemical physics, fluid dynamics, electrical systems, viscoelastic materials, and porous media, thus attracting significant interest from the scientific community [1, 2]. Recently, researchers have been exploring the solvability of linear initial FDE with respect to specific functions in different contexts, where the potential existence of solutions (including positive solutions) is indicated through the application of the Leray-Schoder theory and the fractional Poincaré theorem [3, 4].

The FDE sequential type represents a captivating category of equations. Instances of investigations conducted on such equations are showcased in references [5, 6].

The investigation of the EU of the sequential system below was prompted by the HIV infection model, as discussed in reference [7]:

$$\begin{cases} \left(\mathfrak{D}^{\alpha_1} + k_1\mathfrak{D}^{\alpha_1-1}\right)\upsilon_1(\theta) = \mathfrak{F}_1\left(\theta, \upsilon_1(\theta), \upsilon_1(\theta)\right) \\ \left(\mathfrak{D}^{\alpha_2} + k_2\mathfrak{D}^{\alpha_2-1}\right)\upsilon_2(\theta) = \mathfrak{F}_2\left(\theta, \upsilon_1(\theta), \upsilon_1(\theta)\right) \\ \upsilon_1(0) = \upsilon_1'(0) = 0, \quad \upsilon_1(1) = a\upsilon_2(\xi) \\ \upsilon_2(0) = \upsilon_2'(0) = 0, \quad \upsilon_2(1) = b\upsilon_1(\eta) \end{cases}$$

Corresponding Author: Rahmat.Darzi@iau.ac.ir

https://doi.org/10.48314/anowa.v2i1.67

given $\alpha_1, \alpha_2 \in (2, 3])$, $\theta, \xi, \eta \in (0, 1)$, $k_1, k_2 \in \mathbb{R}^+$, $a, b \in \mathbb{R}$, $\mathfrak{D}^{\alpha_1}$ and $\mathfrak{D}^{\alpha_2}$ represent the Liouville-Caputo sense, while $\nu_1, \nu_2 : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ denote the specified continuous functions. The key methodologies employed in deriving the outcomes involve two fundamental theorems in FP viewpoint: Banach's contraction principle and Leray-Schauder alternative.

A distinct class of issues that has garnered significant interest pertains to the antiperiodic boundary problems encountered in the mathematical representation of specific physical phenomena and occurrences. Notably, extensive investigations have been conducted on the fractional variation of such problems by numerous scholars [8, 9]. An illustration of this can be found in the subsequent nonlinear antiperiodic boundary value problems addressed by the authors in [10].

Many scholars within this discipline contend that employing integral boundary conditions is a more rational approach compared to local boundary conditions. Integral boundary conditions are commonly utilized in various models such as population dynamics, blood flow modeling, heat transmission, and cellular systems.

Previous investigations on Fractional Differential Equations (FDE) and Partial Differential Equations (PDE) with integral boundary conditions can be located in references [11, 12].

In light of the existing literature and addressing the inquiry on the unification of antiperiodic and integral conditions in a system, this paper introduces the utilization of anti-periodic and integral conditions for sequential FDE involving the Liouville-Caputo-type derivative with an order of $2 < \alpha \leq 3$. Additionally, novel existence outcomes that have not been previously presented are provided in this work.

Now consider the following problem:

$$\begin{cases} \mathfrak{D}^{\alpha}\upsilon(\theta) + k\mathfrak{D}^{\alpha-1}\upsilon(\theta) = \mathfrak{F}\left(\theta, \upsilon(\theta), \mathfrak{D}^{\alpha-1}\upsilon(\theta)\right), \\ \beta_1\upsilon(0) + \psi_1\upsilon(1) + \gamma_1\mathfrak{I}^r\upsilon(\varsigma) = \epsilon_1, \\ \beta_2\upsilon'(0) + \psi_2\upsilon'(1) + \gamma_2\mathfrak{I}^r\upsilon'(\varsigma) = \epsilon_2, \\ \beta_3\upsilon''(0) + \psi_3\upsilon''(1) + \gamma_3\mathfrak{I}^r\upsilon''(\varsigma) = \epsilon_3, \end{cases} \tag{1.1}$$

the parameter $\alpha \in (2, 3]$ represents a real number, while $\beta_i, \psi_i, \gamma_i, \epsilon_i \in \mathbb{R}$ for $i = 1, 2, 3$ and $\theta \in [0, 1]$, $k, r, \varsigma > 0$. The operator $\mathfrak{D}^{\alpha}$ denotes the Liouville-Caputo derivative, and the boundary conditions involve antiperiodic cases and R-L fractional integral boundary values. The symbol υ corresponds to a nonlinear term that includes the unknown function. The expression of a linear combination involving the values of an unspecified function and its initial derivatives at the boundaries of a given interval, together featuring the R-L fractional integral value of the aforementioned function and its first and second derivatives at a point within the interval, serve to define the novel constraints placed on the system.

2|Preliminaries

In this part, we introduce the key definitions and lemmas of the theory of integrals and derivatives of arbitrary order that will be referenced in the upcoming sections [13–15].

Definition 1. Let $\mathfrak{F} : [0, \infty) \to \mathbb{R}$ be a function that is continuously differentiable n times. The expression for the Liouville-Caputo fractional derivative of order $\varepsilon > 0$ for $\mathfrak{F}$ can be expressed as

$$\mathfrak{D}^{\varepsilon}\mathfrak{U}(\theta) = \frac{1}{\Gamma(n-\varepsilon)} \int_0^{\theta} (\theta - s)^{n-\varepsilon-1}\mathfrak{U}^{(n)}(s)ds,$$

in which $n \in \mathbb{N}$, $\theta > 0$ and $\varepsilon \in (n-1, n)$.

Lemma 1. The Liouville-Caputo FDE $\mathfrak{D}^{\varepsilon}\mathfrak{U}(\theta) = 0$, featuring $\varepsilon > 0$, possesses the general solution

$$\mathfrak{U}(\theta) = \mathfrak{c}_1 + \mathfrak{c}_2\theta + ... + \mathfrak{c}_n\theta^{n-1},$$

in which $n \in \mathbb{N}$, $\varepsilon \in (n-1, n)$ and $\mathfrak{c}_i \in \mathbb{R}$, $i = 1, 2, ..., n$.

Lemma 2. [16] If $\mathfrak{U} \in C^{n-1}[0, \mathfrak{b})$ and $\mathfrak{D}^{\varepsilon}\mathfrak{U} \in C[0, \mathfrak{b})$, then

$$\mathfrak{I}^{\varepsilon}\mathfrak{D}^{\varepsilon}\mathfrak{U}(\theta) = \mathfrak{U}(\theta) - \sum_{j=0}^{n-1} \frac{\mathfrak{U}^{(j)}(0^+)}{j!}\theta^j,$$

holds on $(0, \mathfrak{b})$ for $n \in \mathbb{N}$ and $\varepsilon \in (n-1, n]$.

NOTATIONS. To facilitate the proof of the subsequent lemma and to further progress the research, the forthcoming symbols will be introduced:

$$\delta_1 = \frac{a_1 + \psi_1}{k} + \frac{\gamma_1\varsigma^r}{k\Gamma(r+1)}, \qquad \delta_2 = \frac{\beta_1 + \psi_1(k+1)}{k^2} + \frac{\gamma_1\varsigma^r(1+k\varsigma)}{k^2\Gamma(r+1)}$$

$$\delta_3 = \beta_1 + \psi_1 e^{-k}\gamma_1 \int_0^{\varsigma} \frac{(\varsigma - s)^{r-1}}{\Gamma(r)} e^{-ks} ds, \qquad \delta_4 = \frac{\beta_2 + \psi_2}{k} + \frac{\gamma_2\varsigma^r}{k\Gamma(r+1)}$$

$$\delta_5 = -k\left(\beta_2 + \psi_2 e^{-k}\gamma_2 \int_0^{\varsigma} \frac{(\varsigma - s)^{r-1}}{\Gamma(r)} e^{-ks} ds\right)$$

$$\delta_6 = k^2\left(\beta_3 + \psi_3 e^{-k}\gamma_3 \int_0^{\varsigma} \frac{(\varsigma - s)^{r-1}}{\Gamma(r)} e^{-ks} ds\right)$$

$$f_1(\theta) = \frac{1}{k\delta_1}, \quad f_2(\theta) = -\frac{1}{k\delta_1} + \frac{1}{k^2\delta_4} + \frac{1}{k\delta_4}\theta,$$

$$f_3(\theta) = -\frac{\delta_2\delta_5 + \delta_1\delta_4}{k\delta_1\delta_4\delta_6} - \frac{\delta_5}{k^2\delta_4\delta_6} - \frac{\delta_5}{k\delta_4\delta_6}\theta + \frac{1}{\delta_6}e^{-k\theta}$$

$$\begin{aligned}\phi_1(\nu(1), \nu(\varsigma)) &= -\psi_1 \int_0^1 e^{-k(1-s)} \left(\mathfrak{I}^{\alpha-1}\nu(s)\right) ds \\ &\quad - \gamma_1 \int_0^{\varsigma} \frac{(\varsigma - s)^{r-1}}{\Gamma(r)} \left(\int_0^s e^{-k(s-m)} \left(\mathfrak{I}^{\alpha-1}\nu(m)\right) dm\right) ds + \epsilon_1\end{aligned}$$

$$\begin{aligned}\phi_2(\nu(1), \nu(\varsigma)) &= -\psi_2 \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\nu(s)ds + k\psi_2 \int_0^1 e^{-k(1-s)} \left(\mathfrak{I}^{\alpha-1}\nu(s)\right) ds \\ &\quad - \gamma_2 \int_0^{\varsigma} \frac{(\varsigma - s)^{r-1}}{\Gamma(r)} \left(\mathfrak{I}^{\alpha-1}\nu(s)\right) ds \\ &\quad + k\gamma_2 \int_0^{\varsigma} \frac{(\varsigma - s)^{r-1}}{\Gamma(r)} \left(\int_0^s e^{-k(s-m)} \left(\mathfrak{I}^{\alpha-1}\nu(m)\right) dm\right) ds + \epsilon_2\end{aligned}$$

$$\phi_3(\nu(1), \nu(\varsigma)) = -\psi_3 \int_0^1 \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}\nu(s)ds + k\psi_3 \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\nu(s)ds$$

$$
\begin{aligned}
&- k^2\psi_3 \int_0^1 e^{-k(1-s)} \left(\mathfrak{I}^{\alpha-1}\nu(s)\right) ds + \gamma_3 \int_0^{\varsigma} \frac{(\varsigma - s)^{r-1}}{\Gamma(r)} \left(\mathfrak{I}^{\alpha-2}\nu(s)\right) ds \\
&\quad + k\gamma_3 \int_0^{\varsigma} \frac{(\varsigma - s)^{r-1}}{\Gamma(r)} \left(\mathfrak{I}^{\alpha-1}\nu(s)\right) ds \\
&\quad - k^2\gamma_3 \int_0^{\varsigma} \frac{(\varsigma - s)^{r-1}}{\Gamma(r)} \left(\int_0^{s} e^{-k(s-m)} \left(I^{\alpha-1}\nu(m)\right) dm\right) ds + \epsilon_3
\end{aligned}
$$

Lemma 3. Authorizing $\nu \in C[0,1]$ and $\upsilon \in C^2[0,1]$. Therefore, the following sequential FDE

$$
\mathfrak{D}^{\alpha}\upsilon(\theta) + k\mathfrak{D}^{\alpha-1}\upsilon(\theta) = \nu(\theta), \tag{2.1}
$$

for $\theta \in [0,1]$ and $k > 0$ with the boundary conditions

$$
\begin{cases}
\beta_1\upsilon(0) + \psi_1\upsilon(1) + \gamma_1\mathfrak{I}^r\upsilon(\varsigma) = \epsilon_1 \\
\beta_2\upsilon'(0) + \psi_2\upsilon'(1) + \gamma_2\mathfrak{I}^r\upsilon'(\varsigma) = \epsilon_2 \\
\beta_3\upsilon''(0) + \psi_3\upsilon''(1) + \gamma_3\mathfrak{I}^r\upsilon''(\varsigma) = \epsilon_3
\end{cases} \tag{2.2}
$$

has a unique solution

$$
\begin{aligned}
\upsilon(\theta) = f_1(\theta)\phi_1\left(\nu(1), \nu(\varsigma)\right) + f_2(\theta)\phi_2\left(\nu(1), \nu(\varsigma)\right) + f_3(\theta)\phi_3\left(\nu(1), \nu(\varsigma)\right) \\
+ \int_0^{\theta} e^{-k(\theta-s)} \left(\mathfrak{I}^{\alpha-1}\nu(s)\right) ds.
\end{aligned} \tag{2.3}
$$

Proof. Authorizing $\upsilon \in C^2[0,1]$ denote a solution to the BVP referenced in equation 2.1. Given that $\upsilon'' \in C[0,1]$, the definition stated in 2.1 elucidates that $\mathfrak{D}^{\alpha-1}\upsilon \in C^1[0,1]$. Furthermore, considering the relationship $\mathfrak{D}^{\alpha}\upsilon = \nu(\theta) - k\mathfrak{D}^{\alpha-1}\upsilon$ and $\nu \in C[0,1]$, it follows that $\mathfrak{D}^{\alpha}\upsilon \in C(0,1)$. Consequently, as per lemma 2.3, the subsequent relations can be deduced

$$
\mathfrak{I}^{\alpha}\mathfrak{D}^{\alpha}\upsilon(\theta) = \upsilon(\theta) - \beta_1 - \beta_2\theta - \beta_3\theta^2, \qquad \theta \in C(0,1), \tag{2.4}
$$

and

$$
\mathfrak{I}^{\alpha-1}\mathfrak{D}^{\alpha-1}\upsilon(\theta) = \upsilon(\theta) - \psi_2 - \psi_3\theta, \qquad \theta \in C(0,1),
$$

So,

$$
\mathfrak{I}^{\alpha}\mathfrak{D}^{\alpha-1}\upsilon(\theta) = \mathfrak{I}^{1}\mathfrak{I}^{\alpha-1}\mathfrak{D}^{\alpha-1}\upsilon(\theta) = \int_0^{\theta} \upsilon(s)ds - \psi_1 - \psi_2\theta - \psi_3\frac{\theta^2}{2}. \tag{2.5}
$$

Here, from 2.1, 2.4 and 2.5, we can gain

$$
\upsilon(\theta) + k\int_0^{\theta} \upsilon(s)ds = c_0 + c_1\theta + c_2\frac{\theta^2}{2} + \mathfrak{I}^{\alpha}\nu(\theta),
$$

in which $c_0, c_1, c_2 \in \mathbb{R}$. It is readily apparent that

$$
\upsilon'(\theta) + k\upsilon(\theta) = c_1 + c_2\theta + \mathfrak{I}^{\alpha-1}\nu(\theta),
$$

and

$$\upsilon''(\theta) + k\upsilon'(\theta) = c_2 + \mathfrak{I}^{\alpha-2}\nu(\theta).$$

In this way, we can come up with the general solution of the FDE as such:

$$\upsilon(\theta) = c_3 e^{-k\theta} + \frac{c_1}{k} + c_2\left(\frac{-1}{k^2} + \frac{\theta}{k}\right) + \int_0^\theta e^{-k(\theta-s)}\left(\mathfrak{I}^{\alpha-1}\nu(s)\right) ds, \tag{2.6}$$

where $c_1, c_2, c_3 \in \mathbb{R}$. Moreover, from this we have

$$\upsilon'(\theta) = -kc_3 e^{-k\theta} + \frac{c_2}{k} + \mathfrak{I}^{\alpha-1}\nu(\theta) - k\int_0^\theta e^{-k(\theta-s)}\left(\mathfrak{I}^{\alpha-1}\nu(s)\right) ds, \tag{2.7}$$

and

$$\upsilon''(\theta) = k^2 c_3 e^{-k\theta} + \mathfrak{I}^{\alpha-2}\nu(\theta) - k\mathfrak{I}^{\alpha-1}\nu(\theta) + k^2\int_0^\theta e^{-k(\theta-s)}\left(\mathfrak{I}^{\alpha-1}\nu(s)\right) ds. \tag{2.8}$$

Utilizing the boundary condition 2.2 within the equations 2.6-2.8 yields

$$\begin{cases} \delta_1 c_1 + \delta_2 c_2 + \delta_3 c_3 = \phi_1\left(\nu(1), \nu(\varsigma)\right) \\ \delta_4 c_2 + \delta_5 c_3 = \phi_2\left(\nu(1), \nu(\varsigma)\right) \\ \delta_6 c_3 = \phi_3\left(\nu(1), \nu(\varsigma)\right). \end{cases} \tag{2.9}$$

A concurrent resolution of the system referenced as 2.9 results in

$$\begin{aligned} c_1 &= \frac{1}{\delta_1}\phi_1\left(\nu(1), \nu(\varsigma)\right) - \frac{\delta_2}{\delta_1\delta_4}\phi_2\left(\nu(1), \nu(\varsigma)\right) - \frac{\delta_2\delta_5 + \delta_3\delta_4}{\delta_1\delta_4\delta_6}\phi_3\left(\nu(1), \nu(\varsigma)\right) \\ c_2 &= \frac{1}{\delta_4}\phi_2\left(\nu(1), \nu(\varsigma)\right) - \frac{\delta_5}{\delta_4\delta_6}\phi_3\left(\nu(1), \nu(\varsigma)\right) \\ c_3 &= \frac{1}{\delta_6}\phi_3\left(\nu(1), \nu(\varsigma)\right). \end{aligned}$$

Replacing c_1, c_2 and c_3 to 2.6, we can get the desirable solution 2.3. The converse form of the lemma is derived through straightforward calculation. The outcome is acquired.

Lemma 4. Authorizing that $\nu \in C([0,1], \mathbb{R})$. So, we can gain

i. $|\phi_1(\nu(1), \nu(\varsigma))| \leq \underbrace{\left(|\psi_1|\frac{1-e^{-k}}{k\Gamma(\alpha)} + |\gamma_1|\frac{\varsigma^{\alpha+r}(1-e^{-k\varsigma})}{k\Gamma(\alpha)\Gamma(r+1)}\right)}_{L_1}\|\nu\| + |\epsilon_1| = L_1\|\nu\| + |\epsilon_1|$

ii. $|\phi_2(\nu(1), \nu(\varsigma))| \leq \underbrace{\left(|\psi_2|\frac{k+1-e^{-k}}{k\Gamma(\alpha)} + |\gamma_2|\frac{\varsigma^{\alpha+r}(k+1-e^{-k\varsigma})}{k\Gamma(\alpha)\Gamma(r+1)}\right)}_{L_2}\|\nu\| + |\epsilon_2|$

$= L_2\|\nu\| + |\epsilon_2|$

iii. $|\phi_3(\nu(1), \nu(\varsigma))| \leq \underbrace{\left(|\psi_3|\frac{\alpha-1+k(2-e^{-k})}{k\Gamma(\alpha)} + |\gamma_3|\frac{(\alpha-1)\varsigma^{\alpha+r-1} + \varsigma^{\alpha+r}(2-e^{-k\varsigma})}{\Gamma(\alpha)\Gamma(r+1)}\right)}_{L_3} \times$

$\|\nu\| + |\epsilon_3| = L_3\|\nu\| + |\epsilon_3|$

Proof. Obviously, we have

$$
\begin{aligned}
&\left|\mathfrak{I}^{\alpha-1}\nu(s)\right| = \frac{1}{\Gamma(\alpha-1)}\int_0^s (s-m)^{\alpha-1}\nu(m)dm = \frac{s^\alpha}{\Gamma(\alpha)}\|\nu\|,\\
&\left|\mathfrak{I}^{\alpha-1}\nu(1)\right| \le \frac{1}{\Gamma(\alpha)}\|\nu\|,\\
&\left|\int_0^s e^{-k(s-m)}\left(\mathfrak{I}^{\alpha-1}\nu(m)\right)dm\right| \le \frac{s^\alpha}{\Gamma(\alpha)}\frac{1-e^{-ks}}{k}\|\nu\|,\\
&\left|\int_0^1 e^{-k(1-s)}\left(\mathfrak{I}^{\alpha-1}\nu(s)\right)ds\right| \le \frac{1}{k\Gamma(\alpha)}\frac{1-e^{-k}}{k}\|\nu\|,\\
&\left|\int_0^\varsigma \frac{(\varsigma-s)^{r-1}}{\Gamma(r)}\left(\mathfrak{I}^{\alpha-1}\nu(s)\right)ds\right| \le \frac{\varsigma^{\alpha+r}}{\Gamma(\alpha)\Gamma(r+1)}\|\nu\|,\\
&\left|\int_0^\varsigma \frac{(\varsigma-s)^{r-1}}{\Gamma(r)}\left(\int_0^s e^{-k(s-m)}\left(\mathfrak{I}^{\alpha-1}\nu(s)\right)dm\right)ds\right| \le \frac{\varsigma^{\alpha+r}(1-e^{-k\varsigma})}{k\Gamma(\alpha)\Gamma(r+1)}\|\nu\|,
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|\phi_1(\nu(1),\nu(\varsigma))| &\le \left|-\psi_1\int_0^1 e^{-k(1-s)}\left(\mathfrak{I}^{\alpha-1}\nu(s)\right)ds\right|\\
&\quad+\left|\gamma_1\int_0^\varsigma \frac{(\varsigma-s)^{r-1}}{\Gamma(r)}\left(\int_0^s e^{-k(s-m)}\left(\mathfrak{I}^{\alpha-1}\nu(s)\right)dm\right)ds+\epsilon_1\right|\\
&\le \left(|\psi_1|\frac{1-e^{-k}}{k\Gamma(\alpha)}+|\gamma_1|\frac{\varsigma^{\alpha+r}(1-e^{-k\varsigma})}{k\Gamma(\alpha)\Gamma(r+1)}\right)\|\nu\|+|\epsilon_1|\\
&= L_1\|\nu\|+|\epsilon_1|
\end{aligned}
$$

In case the proof (ii) and (iii) looks like (i), it is removed.

Set $C[0,1]$ is all the continuous functions on $[0,1]$. Authorizing $C_{\alpha-1}=\{\upsilon\in C[0,1];\ \mathfrak{D}^{\alpha-1}\upsilon\in C[0,1]\}$. Consider $\mathfrak{E}=(C_{\alpha-1},\|.\|_{\alpha-1})$ displays the Banach space armed via the norm given by

$$
\|\upsilon\|_{\alpha-1}=\sup_{0\le\theta\le1}|\upsilon(\theta)|+\sup_{0\le\theta\le1}|\mathfrak{D}^{\alpha-1}\upsilon(\theta)|=\|\upsilon\|+\|\mathfrak{D}^{\alpha-1}\upsilon\|. \tag{2.10}
$$

In connection with lemma 2.4, when $g(t)$ is replaced by $\mathfrak{F}(\theta,\upsilon(\theta))$ in equation 2.1, the resolution of the issue stated in problem 1.1 transforms into the FP of the operator equation $\upsilon=\mathfrak{P}\upsilon$. Here, the operator $\mathfrak{P}:\mathfrak{E}\to\mathfrak{E}$ can be defined as:

$$
\begin{aligned}
\mathfrak{P}\upsilon(\theta) &= f_1(\theta)\phi_1\left(\mathfrak{F}\left(1,\upsilon(1),\mathfrak{D}^{\alpha-1}\upsilon(1)\right),\mathfrak{F}\left(\varsigma,\upsilon(\varsigma),\mathfrak{D}^{\alpha-1}\upsilon(\varsigma)\right)\right)\\
&\quad+f_2(\theta)\phi_2\left(\mathfrak{F}\left(1,\upsilon(1),\mathfrak{D}^{\alpha-1}\upsilon(1)\right),\mathfrak{F}\left(\varsigma,\upsilon(\varsigma),\mathfrak{D}^{\alpha-1}\upsilon(\varsigma)\right)\right)\\
&\quad+f_3(\theta)\phi_3\left(\mathfrak{F}\left(1,\upsilon(1),\mathfrak{D}^{\alpha-1}\upsilon(1)\right),\mathfrak{F}\left(\varsigma,\upsilon(\varsigma),\mathfrak{D}^{\alpha-1}\upsilon(\varsigma)\right)\right)\\
&\quad+\int_0^\theta e^{-k(\theta-s)}\left(\mathfrak{I}^{\alpha-1}\mathfrak{F}\left(s,\upsilon(s),\mathfrak{D}^{\alpha-1}\upsilon(s)\right)\right)ds
\end{aligned} \tag{2.11}
$$

and

$$\mathfrak{D}^{\alpha-1}\mathfrak{P}\upsilon(\theta) = -\frac{k}{\delta_6}\phi_3\left(\mathfrak{F}\left(1,\upsilon(1),\mathfrak{D}^{\alpha-1}\upsilon(1)\right),\mathfrak{F}\left(\eta,\upsilon(\varsigma),\mathfrak{D}^{\alpha-1}\upsilon(\varsigma)\right)\right)\times$$
$$\int_0^\theta \frac{(t-s)^{2-\alpha}}{\Gamma(2-\alpha)}e^{-ks}ds \tag{2.12}$$
$$+\int_0^\theta \frac{(\theta-s)^{2-\alpha}}{\Gamma(2-\alpha)}\left(\mathfrak{I}^{\alpha-1}\mathfrak{F}\left(s,\upsilon(s),\mathfrak{D}^{\alpha-1}\upsilon(s)\right)\right)ds$$
$$+\int_0^\theta \frac{(\theta-s)^{2-\alpha}}{\Gamma(2-\alpha)}\left(\int_0^s e^{-k(s-m)}\left(\mathfrak{I}^{\alpha-1}\mathfrak{F}\left(m,\upsilon(m),\mathfrak{D}^{\alpha-1}\upsilon(m)\right)\right)dm\right)ds$$

Proposition 1. (see [17]). $\mathfrak{P}:\mathfrak{E}\to\mathfrak{E}$ that is characterized by complete continuity. Authorizing

$$\mathfrak{U}=\{\upsilon\in\mathfrak{E}:\ \upsilon=\lambda\mathfrak{E}\upsilon,\ for\ some\ 0<\lambda<1\},$$

afterwards, either set $\mathfrak{U}$ is unbounded or $\mathfrak{P}$ possesses at least one FP

Proposition 2. (see [18]). Let $\mathfrak{E}$ be a Banach space, $\mathfrak{D}\subseteq\mathfrak{E}$ be closed and $\mathfrak{P}:\mathfrak{D}\to\mathfrak{D}$ a strict contraction, i.e., $|\mathfrak{P}\upsilon_2-\mathfrak{P}\upsilon_1|\le k|\upsilon_2-\upsilon_1|$ for some $k\in(0,1)$ and all $\upsilon_1,\upsilon_2\in\mathfrak{D}$. Then, $\mathfrak{P}$ has a unique FP.

3|Main Results

Our hypothesis regarding $\mathfrak{F}$ will be elucidated prior to commencing and presenting the main results

(a) $\mathfrak{F}:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous.

(b) there exist constants $a_{11},a_{12},a_{13}\in\mathbb{R}^+$ such that for all $\theta\in[0,1]$ and $\upsilon,\ \upsilon^*\in\mathbb{R}$:
$|\mathfrak{F}(\theta,\upsilon,\upsilon^*)|\le a_{11}|\upsilon|^{\sigma_1}+a_{12}|\upsilon^*|^{\sigma_2}+a_{13},\quad 0<\sigma_1,\sigma_2<1.$

(c) there exist constants $a_{21},a_{22}\in\mathbb{R}^+$ such that for all $\theta\in[0,1]$ and $\upsilon_1,\upsilon_2,\upsilon_1^*,\upsilon_2^*\in\mathbb{R}:$
$|\mathfrak{F}(\theta,\upsilon_2,\upsilon_2^*)-\mathfrak{F}(\theta,\upsilon_1,\upsilon_1^*)|\le a_{21}|\upsilon_2-\upsilon_1|+a_{22}|\upsilon_2^*-\upsilon_1^*|.$

Theorem 3.1. Suppose that conditions (**a**) and (**b**) are satisfied. Subsequently, it follows that the problem referenced as 1.1 possesses at least one solution.

Proof. Firstly, the characterization of a ball within the space $\mathfrak{E}$ denoted by $\psi_R=\{\upsilon|\ \upsilon\in\mathfrak{E};\|\upsilon\|\le R\}$, is explicitly provided, in which

$$R\ge\max\left\{(3L_4a_{11})^{\frac{1}{1-\sigma_1}},(3L_4a_{12})^{\frac{1}{1-\sigma_2}},3\left(3L_4a_{13}+L_5\right)\right\},$$

$L_4=\sum_{i=1}^{3}M_iL_i+M_4L_3+\frac{\Gamma(\alpha)+2k}{k\Gamma(\alpha)\Gamma(4-\alpha)}$ and $L_5=\sum_{i=1}^{2}M_i|\epsilon_i|+(M_3+M_4)|\epsilon_3|$. The mapping $\mathfrak{P}:\psi_R\to\psi_R$ is demonstrated. When considering $\upsilon\in\psi_R$, application of lemma 2.5 and the constraint (**b**) yields

$$|\mathfrak{P}\upsilon(\theta)|=|f_1(\theta)|\left|\phi_1\left(\mathfrak{F}\left(1,\upsilon(1),\mathfrak{D}^{\alpha-1}\upsilon(1)\right),\mathfrak{F}\left(\varsigma,\upsilon(\varsigma),\mathfrak{D}^{\alpha-1}\upsilon(\varsigma)\right)\right)\right|$$

$$
\begin{aligned}
&+ |f_2(\theta)| \left|\phi_2\left(\mathfrak{F}\left(1, \upsilon(1), \mathfrak{D}^{\alpha-1}\upsilon(1)\right), \mathfrak{F}\left(\varsigma, \upsilon(\varsigma), \mathfrak{D}^{\alpha-1}\upsilon(\varsigma)\right)\right)\right| \\
&+ |f_3(\theta)| \left|\phi_3\left(\mathfrak{F}\left(1, \upsilon(1), \mathfrak{D}^{\alpha-1}\upsilon(1)\right), \mathfrak{F}\left(\varsigma, \upsilon(\varsigma), \mathfrak{D}^{\alpha-1}\upsilon(\varsigma)\right)\right)\right| \\
&+ \int_0^\theta e^{-k(\theta-s)} \left|\mathfrak{I}^{\alpha-1}\mathfrak{F}\left(s, \upsilon(s), \mathfrak{D}^{\alpha-1}\upsilon(s)\right)\right| ds \\
\leq\ & M_1 \left\{\left(|\psi_1|\frac{1-e^{-k}}{k\Gamma(\alpha)} + |\gamma_1|\frac{\varsigma^{\alpha+r}(1-e^{-k\varsigma})}{k\Gamma(\alpha)\Gamma(r+1)}\right)\left(a_{11}R^{\sigma_1} + a_{12}R^{\sigma_2} + a_{13}\right)\right\} + M_1|\epsilon_1| \\
+\ & M_2 \left\{\left(|\psi_2|\frac{k+1-e^{-k}}{k\Gamma(\alpha)} + |\gamma_2|\frac{\varsigma^{\alpha+r}(k+1-e^{-k\varsigma})}{k\Gamma(\alpha)\Gamma(r+1)}\right)\left(a_{11}R^{\sigma_1} + a_{12}R^{\sigma_2} + a_{13}\right)\right\} \\
+\ & M_2|\epsilon_2| + M_3\left\{\left(|\psi_3|\frac{\alpha-1+k(2-e^{-k})}{\Gamma(\alpha)} + |\gamma_3|\frac{(\alpha-1)\varsigma^{\alpha+r-1} + k\varsigma^{\alpha+r}(2-e^{-k\varsigma})}{\Gamma(\alpha)\Gamma(r+1)}\right)\right\} \\
&\times \left(a_{11}R^{\sigma_1} + a_{12}R^{\sigma_2} + a_{13}\right) \\
&+ M_3|\epsilon_3| + \frac{1-e^{-k}}{k\Gamma(\alpha)}\left(a_{11}R^{\sigma_1} + a_{12}R^{\sigma_2} + a_{13}\right) \\
\leq\ & \left(\sum_{i=1}^3 M_i L_i + \frac{1}{k\Gamma(\alpha)}\right)\left(a_{11}R^{\sigma_1} + a_{12}R^{\sigma_2} + a_{13}\right) + \sum_{i=1}^3 M_i|\epsilon_i|
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
\left|\mathfrak{D}^{\alpha-1}\mathfrak{P}\upsilon(\theta)\right| &= \left|\frac{-k}{\delta_6}\phi_3\left(\mathfrak{F}\left(1, \upsilon(1), \mathfrak{D}^{\alpha-1}\upsilon(1)\right), \mathfrak{F}\left(\varsigma, \upsilon(\varsigma), \mathfrak{D}^{\alpha-1}\upsilon(\varsigma)\right)\right)\int_0^\theta \frac{(\theta-s)^{2-\alpha}}{\Gamma(3-\alpha)}e^{-ks}ds\right| \\
&+ \left|\int_0^\theta \frac{(\theta-s)^{2-\alpha}}{\Gamma(3-\alpha)}\mathfrak{I}^{\alpha-1}\mathfrak{F}\left(s, \upsilon(s), \mathfrak{D}^{\alpha-1}\upsilon(s)\right) ds\right| \\
&+ \left|\int_0^\theta \frac{(\theta-s)^{2-\alpha}}{\Gamma(3-\alpha)}\left(\int_0^s e^{-k(s-m)}\mathfrak{I}^{\alpha-1}\mathfrak{F}\left(s, \upsilon(s), \mathfrak{D}^{\alpha-1}\upsilon(s)\right) dm\right) ds\right| \\
&\leq \frac{k}{|\delta_6|}\left\{|\psi_3|\frac{|\alpha-1| + k(2-e^{-k})}{\Gamma(\alpha)} + |\gamma_3|\frac{|\alpha-1|\varsigma^{\alpha+r-1} + k\varsigma^{\alpha+r}(2-e^{-k\varsigma})}{\Gamma(\alpha)\Gamma(r+1)}\right\} \\
&\quad \times \max_{0\leq\theta\leq1}\mathfrak{F}\left(\theta, \upsilon(\theta), \mathfrak{D}^{\alpha-1}\upsilon(\theta)\right)\frac{\theta^{3-\alpha}}{\Gamma(4-\alpha)} \\
&+ \frac{k|\epsilon_3|}{|\delta_6|}\frac{\theta^{3-\alpha}}{\Gamma(4-\alpha)} + \max_{0\leq\theta\leq1}\mathfrak{F}\left(\theta, \upsilon(\theta), \mathfrak{D}^{\alpha-1}\upsilon(\theta)\right)\frac{\theta^{3-\alpha}}{\Gamma(\alpha)\Gamma(4-\alpha)} \\
&+ \max_{0\leq\theta\leq1}\mathfrak{F}\left(\theta, \upsilon(\theta), \mathfrak{D}^{\alpha-1}\upsilon(\theta)\right)\frac{(1-e^{-ks})\theta^{3-\alpha}}{\Gamma(\alpha)\Gamma(4-\alpha)} \\
&\leq \frac{k}{|\delta_6|\Gamma(4-\alpha)}L_3\left(a_{11}R^{\sigma_1} + a_{12}R^{\sigma_2} + a_{13}\right)\frac{k}{|\delta_6|\Gamma(4-\alpha)}|\epsilon_3| \\
&+ \frac{2}{\Gamma(\alpha)\Gamma(4-\alpha)}\left(a_{11}R^{\sigma_1} + a_{12}R^{\sigma_2} + a_{13}\right) \\
&\leq \left(M_4L_3 + \frac{2}{\Gamma(\alpha)\Gamma(4-\alpha)}\right)\left(a_{11}R^{\sigma_1} + a_{12}R^{\sigma_2} + a_{13}\right) + M_4|\epsilon_3|,
\end{aligned}
$$

where $M_4 = \frac{k}{|\delta_6|\Gamma(4-\alpha)}$. Now, the above discussion yields

$$\begin{aligned}
\|\mathfrak{P}\upsilon\|_{\alpha-1} &= \|\mathfrak{P}\upsilon\| + \|\mathfrak{D}^{\alpha-1}\mathfrak{P}\upsilon\| \\
&\leq \left(\sum_{i=1}^{3} M_i L_i + \frac{1}{k\Gamma(\alpha)}\right)(a_{11}R^{\sigma_1} + a_{12}R^{\sigma_2} + a_{13}) + \sum_{i=1}^{3} M_i|\epsilon_i| \\
&\quad + \left(M_4 L_3 + \frac{2}{\Gamma(\alpha)\Gamma(4-\alpha)}\right)(a_{11}R^{\sigma_1} + a_{12}R^{\sigma_2} + a_{13}) + M_4|\epsilon_3| \\
&= \left(\sum_{i=1}^{3} M_i L_i + M_4 L_3 + \frac{\Gamma(4-\alpha)+2k}{k\Gamma(\alpha)\Gamma(4-\alpha)}\right)(a_{11}R^{\sigma_1} + a_{12}R^{\sigma_2} + a_{13}) \\
&\quad + \sum_{i=1}^{2} M_i|\epsilon_i| + (M_3 + M_4)|\epsilon_3| \\
&= L_4 a_{11} R^{\sigma_1} + L_4 a_{12} R^{\sigma_2} + L_4 a_{13} + L_5 \\
&\leq \frac{R}{3} + \frac{R}{3} + \frac{R}{3}.
\end{aligned}$$

This signifies the mapping $\mathfrak{P} : \psi_R \to \psi_R$. By referring to equations (2.16) and (2.17), one can comprehend that the operator $\mathfrak{P}$ exhibits continuity on the interval $[0, 1]$. Subsequently, it will be demonstrated that $\mathfrak{P}$ represents an equicontinuous operator.

Authorizing $\theta_1, \theta_2 \in [0, 1]$ *with* $\theta_1 < \theta_2$. Set $M = \max_{\theta\in[0,1]} \left|\mathfrak{F}\left(\theta, \upsilon(\theta), \mathfrak{D}^{\alpha-1}\upsilon(\theta)\right)\right|, \ \forall \upsilon \in \psi_R$. Ergo

$$\begin{aligned}
|\mathfrak{P}\upsilon(\theta_2) - \mathfrak{P}\upsilon(\theta_1)| &= |f_2(\theta_2) - f_2(\theta_1)| \left|\phi_2\left(\mathfrak{F}(1, \upsilon(1), \mathfrak{D}^{\alpha-1}\upsilon(1)), \mathfrak{F}(\varsigma, \upsilon(\varsigma), \mathfrak{D}^{\alpha-1}\upsilon(\varsigma))\right)\right| \\
&\quad + |f_3(\theta_2) - f_3(\theta_1)| \left|\phi_3\left(\mathfrak{F}(1, \upsilon(1), \mathfrak{D}^{\alpha-1}\upsilon(1)), \mathfrak{F}(\varsigma, \upsilon(\varsigma), \mathfrak{D}^{\alpha-1}\upsilon(\varsigma))\right)\right| \\
&\quad + \left|\int_0^{\theta_2} e^{-k(\theta_2-s)}\left(\mathfrak{J}^{\alpha-1}\mathfrak{F}(s, \upsilon(s), \mathfrak{D}^{\alpha-1}\upsilon(s))\right) ds\right. \\
&\quad \left. - \int_0^{\theta_1} e^{-k(\theta_1-s)}\left(\mathfrak{J}^{\alpha-1}\mathfrak{F}\left(s, \upsilon(s), \mathfrak{D}^{\alpha-1}\upsilon(s)\right)\right) ds\right| \\
&\leq \frac{1}{k|\delta_4|}\left(|\psi_2|\frac{k+1-e^{-k}}{k\Gamma(\alpha)} + |\gamma_2|\frac{\varsigma^{\alpha+r}(k+1-e^{-k\varsigma})}{k\Gamma(\alpha)\Gamma(r+1)} + |\epsilon_2|\right)(\theta_2 - \theta_1) \\
&\quad + \frac{1}{|\delta_6|}\left(e^{-k\theta_1} - e^{-k\theta_2}\right) + \frac{|\delta_5|M}{k|\delta_4\delta_6|}(\theta_2 - \theta_1)\times \\
&\quad \left(|\psi_3|\frac{\alpha-1+k(2-e^{-k})}{\Gamma(\alpha)} + |\gamma_3|\frac{(\alpha-1)\varsigma^{\alpha+r-1} + k\varsigma^{\alpha+r}(2-e^{-k\eta})}{\Gamma(\alpha)\Gamma(r+1)}\right) \\
&\quad + \frac{|\delta_5|M}{k|\delta_4\delta_6|}|\epsilon_2|\ (\theta_2 - \theta_1) + \int_{\theta_1}^{\theta_2} e^{-k(\theta_2-s)}\left|\mathfrak{J}^{\alpha-1}\mathfrak{F}(s, \upsilon(s), \mathfrak{D}^{\alpha-1}\upsilon(s))\right| ds \\
&\quad + \int_0^{\theta_1}\left(e^{-k(\theta_1-s)} - e^{-k(\theta_2-s)}\right)\left|\mathfrak{J}^{\alpha-1}\mathfrak{F}(s, \upsilon(s), \mathfrak{D}^{\alpha-1}\upsilon(s))\right| ds \\
&\leq M\left(\frac{L_2}{k|\delta_3|} + \frac{L_3|\delta_5|}{k|\delta_4\delta_6|} + |\epsilon_2| + |\epsilon_3|\right)(\theta_2 - \theta_1) + \frac{1}{|\delta_6|}\left(e^{-k\theta_1} - e^{-k\theta_2}\right)
\end{aligned}$$

$$+\frac{M}{k\Gamma(\alpha)}\left\{2\left(1-e^{-k(\theta_2-\theta_1)}\right)+\left(e^{-k\theta_1}-e^{-k\theta_2}\right)\right\}.$$

Also,we have

$$\begin{aligned}\left|\mathfrak{D}^{\alpha-1}\mathfrak{P}\upsilon(\theta_2)-\mathfrak{D}^{\alpha-1}\mathfrak{P}\upsilon(\theta_1)\right| &= \frac{k}{|\delta_6|}\left|\phi_3\left(\mathfrak{F}\left(1,\upsilon(1),\mathfrak{D}^{\alpha-1}\upsilon(1)\right),\mathfrak{F}\left(\varsigma,\upsilon(\varsigma),\mathfrak{D}^{\alpha-1}\upsilon(\varsigma)\right)\right)\right| \\ &\quad\times\left|\int_0^{\theta_2}\frac{(\theta_2-s)^{2-\alpha}}{\Gamma(3-\alpha)}e^{-ks}ds-\int_0^{\theta_1}\frac{(\theta_1-s)^{2-\alpha}}{\Gamma(3-\alpha)}e^{-ks}ds\right| \\ &\quad+\left|\int_0^{\theta_2}\frac{(\theta_2-s)^{2-\alpha}}{\Gamma(3-\alpha)}\left(\mathfrak{I}^{\alpha-1}\mathfrak{F}\left(s,\upsilon(s),\mathfrak{D}^{\alpha-1}\upsilon(s)\right)\right)ds\right. \\ &\quad\left.-\int_0^{\theta_1}\frac{(\theta_1-s)^{2-\alpha}}{\Gamma(3-\alpha)}\left(\mathfrak{I}^{\alpha-1}\mathfrak{F}\left(s,\upsilon(s),\mathfrak{D}^{\alpha-1}\upsilon(s)\right)\right)ds\right| \\ &\quad+k\left|\int_0^{\theta_2}\frac{(\theta_2-s)^{2-\alpha}}{\Gamma(3-\alpha)}\left(\mathfrak{I}^{\alpha-1}\mathfrak{F}\left(s,\upsilon(s),\mathfrak{D}^{\alpha-1}\upsilon(s)\right)\right)ds\right. \\ &\quad\left.-\int_0^{\theta_1}\frac{(\theta_1-s)^{2-\alpha}}{\Gamma(3-\alpha)}\left(\int_0^{s}e^{-k(s-m)}\left(\mathfrak{I}^{\alpha-1}\mathfrak{F}\right)dm\right)ds\right| \\ &\leq\left(\frac{k(ML_3+|\epsilon_3|)e^{-ks}}{|\delta_6|}+\frac{M(2-e^{-ks})}{k\Gamma(\alpha)}\right)\times \\ &\quad\left(\frac{\theta_2^{3-\alpha}-\theta_1^{3-\alpha}+2(\theta_2-\theta_1)^{3-\alpha}}{\Gamma(4-\alpha)}\right).\end{aligned}$$

Assuming $\theta_1\to\theta_2$, then $|\mathfrak{P}\upsilon(\theta_2)-\mathfrak{P}\upsilon(\theta_1)|\to 0$, and

$$\left|\mathfrak{D}^{\alpha-1}\mathfrak{P}\upsilon(\theta_2)-\mathfrak{D}^{\alpha-1}\mathfrak{P}\upsilon(\theta_1)\right|\to 0,$$

that is, as $\theta_1\to\theta_2$,

$$\|\mathfrak{P}\upsilon(\theta_2)-\mathfrak{P}\upsilon(\theta_1)\|_{\alpha-1}\to 0.$$

Hence, the inclusion $\mathfrak{P}(\psi_R)\subseteq\psi_r$ establishes an equicontinuous set. Besides, the set is characterized as uniformly bounded due to $\mathfrak{P}(\psi_R)\subseteq\psi_r$. By employing the Arzela-Ascoli theorem, it is possible to deduce that the operator $\mathfrak{P}$ exhibits complete continuity.

Upon examination of $\mathfrak{U}=\{\upsilon\in\psi_R|\ \upsilon=\mu\mathfrak{P}\upsilon,\ \mu\in(0,1)\}$, it is essential to establish the boundedness of $\mathfrak{U}$. For any $\upsilon\in\mathfrak{U}$, it is known that $\|\upsilon\|_{\alpha-1}<\|\mathfrak{P}\upsilon\|_{\alpha-1}\leq R$. This observation correlates with proposition 2.6 in order to demonstrate that the 1.1 possesses at least one solution within ψ_R, thereby, confirming the proof.

Theorem 1. If assumptions (**a**) and (**c**) are satisfied, and the condition $L_4\left(a_{21}+a_{22}\right)<1$ holds, then the problem stated in reference 1.1 possesses a unique solution.

Proof. Taking $sup_{\theta\in[0,1]}|\mathfrak{F}(\theta,0,0)|=N<\infty$ way that $r\geq\frac{L_4N+L_5}{1-L_4(a_{21}+a_{22})}$. Firstly, it is revealed that $\mathfrak{P}(\psi_r)\subseteq\psi_r$, where $\psi_r=\{\upsilon|\ \upsilon\in\mathfrak{E};\|\upsilon\|_{\alpha-1}\leq r\}$. For each υ belonging to the set ψ_r, through direct computation, it can be shown that

$$|\mathfrak{P}\upsilon(\theta)|=|f_1(\theta)|\left|\phi_1\left(\mathfrak{F}(1,\upsilon(1),\mathfrak{D}^{\alpha-1}\upsilon(1)),\mathfrak{F}(\varsigma,\upsilon(\varsigma),\mathfrak{D}^{\alpha-1}\upsilon(\varsigma))\right)\right|$$

$$
\begin{aligned}
&+ |f_2(\theta)| \left|\phi_2 \left(\mathfrak{F}(1, \upsilon(1), \mathfrak{D}^{\alpha-1}\upsilon(1)), \mathfrak{F}(\varsigma, \upsilon(\varsigma), \mathfrak{D}^{\alpha-1}\upsilon(\varsigma))\right)\right| \\
&+ |f_3(\theta)| \left|\phi_3 \left(\mathfrak{F}(1, \upsilon(1), \mathfrak{D}^{\alpha-1}\upsilon(1)), \mathfrak{F}(\varsigma, \upsilon(\varsigma), \mathfrak{D}^{\alpha-1}\upsilon(\varsigma))\right)\right| \\
&\quad + \int_0^\theta e^{-k(\theta-s)} \left|\mathfrak{I}^{\alpha-1} \left(\mathfrak{F}(s, \upsilon(s), \mathfrak{D}^{\alpha-1}\upsilon(s))\right)\right| ds \\
&\leq M_1 L_1 \left(\max_{0\leq t\leq 1} \left|\mathfrak{F}(\theta, \upsilon(\theta), \mathfrak{D}^{\alpha-1}\upsilon(\theta)) - \mathfrak{F}(\theta, 0, 0) + \mathfrak{F}(\theta, 0, 0)\right|\right) \\
&+ M_1|\epsilon_1| + M_2 L_2 \left(\max_{0\leq t\leq 1} \left|\mathfrak{F}(\theta, \upsilon(\theta), \mathfrak{D}^{\alpha-1}\upsilon(\theta)) - \mathfrak{F}(\theta, 0, 0) + \mathfrak{F}(\theta, 0, 0)\right|\right) \\
&+ M_2|\epsilon_2| + M_3 L_3 \left(\max_{0\leq \theta\leq 1} \left|\mathfrak{F}(\theta, \upsilon(\theta), \mathfrak{D}^{\alpha-1}\upsilon(\theta)) - \mathfrak{F}(\theta, 0, 0) + \mathfrak{F}(\theta, 0, 0)\right|\right) \\
&+ M_3|\epsilon_3| + \left(\max_{0\leq \theta\leq 1} \left|\mathfrak{F}(\theta, \upsilon(\theta), \mathfrak{D}^{\alpha-1}\upsilon(\theta)) - \mathfrak{F}(\theta, 0, 0) + \mathfrak{F}(\theta, 0, 0)\right|\right) \times \\
&\qquad \frac{(1-e^{-k\theta})\theta^\alpha}{k\Gamma(\alpha)} \\
&\leq M_1 L_1 \left(a_{21}\|\upsilon\| + a_{22}\|\mathfrak{D}^{\alpha-1}\upsilon\| + N\right) + M_1|\epsilon_1| \\
&+ M_2 L_2 \left(a_{21}\|\upsilon\| + a_{22}\|\mathfrak{D}^{\alpha-1}\upsilon\| + N\right) + M_2|\epsilon_2| \\
&+ M_3 L_3 \left(a_{21}\|\upsilon\| + a_{22}\|\mathfrak{D}^{\alpha-1}\upsilon\| + N\right) + M_3|\epsilon_3| \\
&+ \frac{1}{k\Gamma(\alpha)} \left(a_{21}\|\upsilon\| + a_{22}\|\mathfrak{D}^{\alpha-1}\upsilon\| + N\right) \\
&\leq \left(M_1 L_1 + M_2 L_2 + M_3 L_3 + \frac{1}{k\Gamma(\alpha)}\right) \left(a_{21}\|\upsilon\| + a_{22}\|\mathfrak{D}^{\alpha-1}\upsilon\| + N\right) \\
&+ \left(M_1|\epsilon_1| + M_2|\epsilon_2| + M_3|\epsilon_3|\right).
\end{aligned}
$$

Also, for any $\upsilon \in \psi_r$, we have

$$
\begin{aligned}
\left|\mathfrak{D}^{\alpha-1}\mathfrak{P}\upsilon(\theta)\right| &\leq \frac{k}{|\delta_6|} \left|\phi_3 \left(\mathfrak{F}(1, \upsilon(1), \mathfrak{D}^{\alpha-1}\upsilon(1)), \mathfrak{F}(\varsigma, \upsilon(\varsigma), \mathfrak{D}^{\alpha-1}\upsilon(\varsigma))\right)\right| \times \\
&\qquad \int_0^\theta \frac{(\theta-s)^{2-\alpha}}{\Gamma(3-\alpha)} e^{-ks} ds \\
&+ \int_0^\theta \frac{(\theta-s)^{2-\alpha}}{\Gamma(3-\alpha)} \left|\mathfrak{I}^{\alpha-1} \left(\mathfrak{F}(s, \upsilon(s), \mathfrak{D}^{\alpha-1}\upsilon(s))\right)\right| ds \\
&+ k \int_0^\theta \frac{(\theta-s)^{2-\alpha}}{\Gamma(3-\alpha)} \left(\int_0^s e^{-k(s-m} \left|\mathfrak{I}^{\alpha-1} \left(\mathfrak{F}(m, \upsilon(m), \mathfrak{D}^{\alpha-1}\upsilon(m))\right)\right| dm\right) ds \\
&\leq \frac{k}{|\delta_6|} \left(L_3 \max_{0\leq \theta\leq 1} \left|\mathfrak{F}(\theta, \upsilon(\theta), \mathfrak{D}^{\alpha-1}\upsilon(\theta)) - \mathfrak{F}(\theta, 0, 0) + \mathfrak{F}(\theta, 0, 0)\right| + |\epsilon_3|\right) \\
&\qquad \times \frac{\theta^{3-\alpha}}{\Gamma(4-\alpha)} \\
&+ \frac{1}{\Gamma(\alpha)} \left(\max_{0\leq \theta\leq 1} \left|\mathfrak{F}(\theta, \upsilon(\theta), \mathfrak{D}^{\alpha-1}\upsilon(\theta)) - \mathfrak{F}(\theta, 0, 0) + \mathfrak{F}(\theta, 0, 0)\right|\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad \times \frac{\theta^{3-\alpha}}{\Gamma(4-\alpha)} \\
&+ \frac{s^{\alpha}(1-e^{-ks})}{\Gamma(\alpha)} \left(\max_{0\le\theta\le 1} \left| \mathfrak{F}(\theta, \upsilon(\theta), \mathfrak{D}^{\alpha-1}\upsilon(\theta)) - \mathfrak{F}(\theta,0,0) + \mathfrak{F}(\theta,0,0) \right| \right) \\
&\quad \times \frac{\theta^{3-\alpha}}{\Gamma(4-\alpha)} \\
&\le \frac{k}{|\delta_6|\Gamma(4-\alpha)} \Big(L_3 \left(a_{21}\|\upsilon\| + a_{22}\|\mathfrak{D}^{\alpha-1}\upsilon\| + N \right) + |\epsilon_3| \Big) \\
&\quad + \frac{1}{\Gamma(\alpha)\Gamma(4-\alpha)} \left(a_{21}\|\upsilon\| + a_{22}\|\mathfrak{D}^{\alpha-1}\upsilon\| + N \right) \\
&+ \frac{1}{\Gamma(\alpha)\Gamma(4-\alpha)} \left(a_{21}\|\upsilon\| + a_{22}\|\mathfrak{D}^{\alpha-1}\upsilon\| + N \right) \\
&= \left(\frac{kL_3}{|\delta_6|\Gamma(4-\alpha)} + \frac{2}{\Gamma(\alpha)\Gamma(4-\alpha)} \right) \left(a_{21}\|\upsilon\| + a_{22}\|\mathfrak{D}^{\alpha-1}\upsilon\| + N \right) \\
&+ \frac{k}{|\delta_6|\Gamma(4-\alpha)} |\epsilon_3|. \\
&= \left(M_4L_3 + \frac{2}{\Gamma(\alpha)\Gamma(4-\alpha)} \right) \left(a_{21}\|\upsilon\| + a_{22}\|\mathfrak{D}^{\alpha-1}\upsilon\| + N \right) + M_4|\epsilon_3|.
\end{aligned}
$$

Now, with the help of above discussion, we acquire

$$
\begin{aligned}
\|\mathfrak{P}\upsilon\|_{\alpha-1} &= \|\mathfrak{P}\upsilon\| + \|\mathfrak{D}^{\alpha-1}\mathfrak{P}\upsilon\| \\
&\le \left(\sum_{i=1}^{3} M_iL_i + \frac{1}{k\Gamma(\alpha)} \right) \left(a_{21}\|\upsilon\| + a_{22}\|\mathfrak{D}^{\alpha-1}\upsilon\| + N \right) + \sum_{i=1}^{3} M_i|\epsilon_i| \\
&\quad + \left(M_4L_3 + \frac{2}{\Gamma(\alpha)\Gamma(4-\alpha)} \right) \left(a_{21}\|\upsilon\| + a_{22}\|\mathfrak{D}^{\alpha-1}\upsilon\| + N \right) + M_4|\epsilon_3| \\
&= \left(\sum_{i=1}^{3} M_iL_i + M_4L_3 + \frac{\Gamma(4-\alpha)+2k}{k\Gamma(\alpha)\Gamma(4-\alpha)} \right) \left(a_{21}\|\upsilon\| + a_{22}\|\mathfrak{D}^{\alpha-1}\upsilon\| + N \right) \\
&\quad + \sum_{i=1}^{2} M_i|\epsilon_i| + (M_3 + M_4)|\epsilon_3| \\
&= L_4 (a_{21} + a_{22}) \|\upsilon\|_{\alpha-1} + L_4N + L_5 \\
&\le r.
\end{aligned}
$$

where $L_4 = \sum_{i=1}^{3} M_iL_i + M_4L_3 + \frac{\Gamma(4-\alpha)+2k}{k\Gamma(\alpha)\Gamma(4-\alpha)}$ and $L_5 = \sum_{i=1}^{2} M_i|\epsilon_i| + (M_3+M_4)|\epsilon_3|$. Also, for any $\upsilon_1, \upsilon_2 \in \psi_r$, we have

$$
\begin{aligned}
|\mathfrak{P}\upsilon_2(\theta) - \mathfrak{P}\upsilon_1(\theta)| &\le |f_1(\theta)| \Big| \phi_1\Big(\mathfrak{F}\left(1, \upsilon_2(1), \mathfrak{D}^{\alpha-1}\upsilon_2(1)\right), \mathfrak{F}\left(\varsigma, \upsilon_2(\varsigma), \mathfrak{D}^{\alpha-1}\upsilon_2(\varsigma)\right) \Big) \\
&\quad - \phi_1\Big(\mathfrak{F}\left(1, \upsilon_1(1), \mathfrak{D}^{\alpha-1}\upsilon_1(1)\right), \mathfrak{F}\left(\varsigma, \upsilon_1(\varsigma), \mathfrak{D}^{\alpha-1}\upsilon_1(\varsigma)\right) \Big) \Big| \\
&\quad + |f_2(\theta)| \Big| \phi_2\Big(\mathfrak{F}\left(1, \upsilon_2(1), \mathfrak{D}^{\alpha-1}\upsilon_2(1)\right), \mathfrak{F}\left(\varsigma, \upsilon_2(\varsigma), \mathfrak{D}^{\alpha-1}\upsilon_2(\varsigma)\right) \Big)
\end{aligned}
$$

$$
\begin{aligned}
&-\phi_1\Big(\mathfrak{F}\left(1, \upsilon_1(1), \mathfrak{D}^{\alpha-1}\upsilon_1(1)\right), \mathfrak{F}\left(\varsigma, \upsilon_1(\varsigma), \mathfrak{D}^{\alpha-1}\upsilon_1(\varsigma)\right)\Big)\Big| \\
&+ |f_3(\theta)| \Big|\phi_3\Big(\mathfrak{F}\left(1, \upsilon_2(1), \mathfrak{D}^{\alpha-1}\upsilon_2(1)\right), \mathfrak{F}\left(\varsigma, \upsilon_2(\varsigma), \mathfrak{D}^{\alpha-1}\upsilon_2(\varsigma)\right)\Big) \\
&-\phi_1\Big(\mathfrak{F}\left(1, \upsilon_1(1), \mathfrak{D}^{\alpha-1}\upsilon_1(1)\right), \mathfrak{F}\left(\varsigma, \upsilon_1(\varsigma), \mathfrak{D}^{\alpha-1}\upsilon_1(\varsigma)\right)\Big)\Big| \\
&- \int_0^t e^{-k(t-s)}\Big(\mathfrak{I}^{\alpha-1}\left|\mathfrak{F}\left(s, \upsilon_2(s), \mathfrak{D}^{\alpha-1}\upsilon_2(s)\right) - \mathfrak{F}\left(s, \upsilon_1(s), \mathfrak{D}^{\alpha-1}\upsilon_1(s)\right)\right|\Big)ds \\
&\le \left(M_1L_1 + M_2L_3 + M_3L_3 + \frac{1}{k\Gamma(\alpha)}\right)\left(a_{21}\|\upsilon_2 - \upsilon_1\| + a_{22}\|\mathfrak{D}^{\alpha-1}\upsilon_2 - \mathfrak{D}^{\alpha-1}\upsilon_1\|\right) \\
&\le \left(\sum_{i=1}^{3} M_iL_i + \frac{1}{k\Gamma(\alpha)}\right)(a_{21} + a_{22})\|\upsilon_2 - \upsilon_1\|_{\alpha-1}.
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\mathfrak{D}^{\alpha-1}\mathfrak{P}\upsilon_2(\theta) - \mathfrak{D}^{\alpha-1}\mathfrak{P}\upsilon_1(\theta)\right| \le{}& \frac{k}{|\delta_6|}\Big|\phi_3\Big(\mathfrak{F}\left(1, \upsilon_2(1), \mathfrak{D}^{\alpha-1}\upsilon_2(1)\right), \mathfrak{F}\left(\varsigma, \upsilon_2(\varsigma), \mathfrak{D}^{\alpha-1}\upsilon_2(\varsigma)\right)\Big) \\
&-\phi_3\Big(\mathfrak{F}\left(1, \upsilon_1(1), \mathfrak{D}^{\alpha-1}\upsilon_1(1)\right), \mathfrak{F}\left(\varsigma, \upsilon_1(\varsigma), \mathfrak{D}^{\alpha-1}\upsilon_1(\varsigma)\right)\Big)\Big| \\
&\times \int_0^\theta \frac{(\theta - s)^{2-\alpha}}{\Gamma(3-\alpha)} e^{-ks} ds \\
&+ \int_0^\theta \frac{(\theta - s)^{2-\alpha}}{\Gamma(3-\alpha)} \\
&\times \Big(\mathfrak{I}^{\alpha-1}\left|\mathfrak{F}\left(s, \upsilon_2(s), \mathfrak{D}^{\alpha-1}\upsilon_2(s)\right) - \mathfrak{F}\left(s, \upsilon_1(s), \mathfrak{D}^{\alpha-1}\upsilon_1(s)\right)\right|\Big)ds \\
&+ \int_0^\theta \frac{(\theta - s)^{2-\alpha}}{\Gamma(3-\alpha)} \Big(\int_0^s e^{k(s-m)} \\
&\times\Big(\mathfrak{I}^{\alpha-1}\left|\mathfrak{F}\left(m, \upsilon_2(m), \mathfrak{D}^{\alpha-1}\upsilon_2(m)\right) - \mathfrak{F}\left(m, \upsilon_1(m), \mathfrak{D}^{\alpha-1}\upsilon_1(m)\right)\right|\Big)dm\Big)\, ds \\
&\le \frac{kL_3}{|\delta_6|\Gamma(4-\alpha)}\left(a_{21}\|\upsilon_2 - \upsilon_1\| + a_{22}\|\mathfrak{D}^{\alpha-1}\upsilon_2 - \mathfrak{D}^{\alpha-1}\upsilon_1\|\right) \\
&+ \frac{1}{|\Gamma(\alpha)\Gamma(4-\alpha)}\left(a_{21}\|\upsilon_2 - \upsilon_1\| + a_{22}\|\mathfrak{D}^{\alpha-1}\upsilon_2 - \mathfrak{D}^{\alpha-1}\upsilon_1\|\right) \\
&+ \frac{1 - e^{-k\theta}}{|\Gamma(\alpha)\Gamma(4-\alpha)}\left(a_{21}\|\upsilon_2 - \upsilon_1\| + a_{22}\|\mathfrak{D}^{\alpha-1}\upsilon_2 - \mathfrak{D}^{\alpha-1}\upsilon_1\|\right) \\
&\le \left(\frac{kL_3}{|\delta_6|\Gamma(4-\alpha)} + \frac{2}{|\Gamma(\alpha)\Gamma(4-\alpha)}\right) \\
&\quad\times\left(a_{21}\|\upsilon_2 - \upsilon_1\| + a_{22}\|\mathfrak{D}^{\alpha-1}\upsilon_2 - \mathfrak{D}^{\alpha-1}\upsilon_1\|\right) \\
&\le \left(M_4L_3 + \frac{2}{|\Gamma(\alpha)\Gamma(4-\alpha)}\right)(a_{21} + a_{22})\|\upsilon_2 - \upsilon_1\|_{\alpha-1}.
\end{aligned}
$$

Therefore

$$
\|\mathfrak{P}\upsilon_2 - \mathfrak{P}\upsilon_1\|_{\alpha-1} = \|\mathfrak{P}\upsilon_2 - \mathfrak{P}\upsilon_1\| + \|\mathfrak{D}^{\alpha-1}\mathfrak{P}\upsilon_2 - \mathfrak{D}^{\alpha-1}\mathbb{F}\mathfrak{P}\upsilon_1\|
$$

$$
\begin{aligned}
&\leq \left(\sum_{i=1}^{3} M_i L_i + M_4 L_3 + \frac{\Gamma(4-\alpha)+2k}{k\Gamma(\alpha)\Gamma(4-\alpha)}\right)(a_{21}+a_{22})\,\|\upsilon_2-\upsilon_1\|_{\alpha-1} \\
&= L_4\,(a_{21}+a_{22})\,\|\upsilon_2-\upsilon_1\|_{\alpha-1}.
\end{aligned}
$$

Given that $L_4\,(a_{21}+a_{22}) < 1$, it can be deduced that the operator $\mathfrak{P}$ is constrained. This conclusion is drawn from proposition 2.7. Consequently, $\mathfrak{P}$ possesses a distinct FP, implying that the system 1.1 boasts a unique solution. This observation effectively shows that the result has been achieved.

4|Examples

Now that the investigation into the outcomes of the EU of the solution has been successful, two illustrative examples are presented in order to demonstrate the efficacy of the acquired results in the present study. Assume

$$
\begin{cases}
\mathfrak{D}^{2.5}\upsilon(\theta) + k\mathfrak{D}^{1.5}\upsilon(\theta) = \mathfrak{F}\left(\theta, \upsilon(\theta), \mathfrak{D}^{1.5}\upsilon(\theta)\right) \\
0.01\upsilon(0) + 0.02\upsilon(1) - \mathfrak{I}^2\upsilon(0.5) = 0.0001 \\
0.02\upsilon'(0) + 0.01\upsilon'(1) + 2\mathfrak{I}^2\upsilon'(0.5) = 0.0002 \\
0.12\upsilon''(0) + 0.18\upsilon''(1) - 3\mathfrak{I}^2\upsilon''(0.5) = 0.0003.
\end{cases}
\tag{4.1}
$$

in which $\theta \in [0,1]$ and $k > 0$. As claimed by problem 1.1, it is easy to understand $\alpha = 2.5, k = 1, r = 2, \varsigma = 0.5, \beta_1 = 0.01, \beta_2 = 0.02, \beta_3 = 0.12,\ \psi_1 = 0.02, \psi_2 = 0.01, \psi_3 = 0.18,\ \gamma_1 = -1, \gamma_2 = 2, \gamma_3 = -3,\ \epsilon_1 = 0.0001, \epsilon_2 = 0.0002, \epsilon_3 = 0.0003$.

Acknowledgments

The authors sincerely thank all those who contributed to this research through their guidance, expertise, and helpful recommendations.

Funding

No funding was received for this research.

Data Availability

Relevant data supporting the conclusions of this study can be obtained from the corresponding author upon reasonable request.

References

[1] Furati, K. M. (2012). A Cauchy-type problem with a sequential fractional derivative in the space of continuous functions. *Boundary value problems*, *2012*(1), 58. https://doi.org/10.1186/1687-2770-2012-58

[2] Hilfer, R. (2000). *Applications of fractional calculus in physics*. World Scientific Publishing. https://www2.icp.uni-stuttgart.de/~hilfer/publikationen/pdfa/applications-of-fractional-calculus-in-physics.pdf

[3] Agarwal, R. P., O'Regan, D., & Staněk, S. (2010). Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations. *Journal of mathematical analysis and applications*, *371*(1), 57-68. https://doi.org/10.1016/j.jmaa.2010.04.034

[4] Bai, Z., & Lü, H. (2005). Positive solutions for boundary value problem of nonlinear fractional differential equation. *Journal of mathematical analysis and applications*, *311*(2), 495-505. https://doi.org/10.1016/j.jmaa.2005.02.052

[5] Ahmad, B., & Luca, R. (2018). Existence of solutions for sequential fractional integro-differential equations and inclusions with nonlocal boundary conditions. *Applied mathematics and computation*, *339*, 516-534. https://doi.org/10.1016/j.amc.2018.07.025

[6] Ahmad, B., & Nieto, J. J. (2012). Sequential fractional differential equations with three-point boundary conditions. *Computers & mathematics with applications*, *64*(10), 3046-3052. https://doi.org/10.1016/j.camwa.2012.02.036

[7] Jiang, J., & Liu, L. (2016). Existence of solutions for a sequential fractional differential system with coupled boundary conditions. *Boundary value problems*, *2016*(1), 159. https://doi.org/10.1186/s13661-016-0666-8

[8] Ahmad, B., Alsaedi, A., & Aqlan, M. H. (2017). Sequential fractional differential equations and unification of anti-periodic and multi-point boundary conditions. *Journal of nonlinear science and applications, 10*, 71-83. http://dx.doi.org/10.22436/jnsa.010.01.07

[9] Luo, Y. (2020). Antiperiodic boundary value problems for first-order impulsive differential inclusions. *Indian journal of pure and applied mathematics*, *51*(4), 1373-1381. https://doi.org/10.1007/s13226-020-0471-z

[10] Alsaedi, A., Aqlan, M. H., & Ahmad, B. (2018). On anti-periodic type boundary value problems of sequential fractional differential equations of order q∈(2, 3]. *Journal of computational analysis & applications*, *24*(2), 310. https://www.researchgate.net/publication/322170877_On_anti-periodic_type_boundary_value_problems_of_sequential_fractional_differential_equations_of_order_q_2_3

[11] Zhang, H., Li, Y., & Lu, W. (2016). Existence and uniqueness of solutions for a coupled system of nonlinear fractional differential equations with fractional integral boundary conditions. *Journal of nonlinear science and applications, 9*(05), 2434-2447. https://doi.org/10.22436/jnsa.009.05.43

[12] Qi, T., Liu, Y., & Zou, Y. (2017). Existence result for a class of coupled fractional differential systems with integral boundary value conditions. *Journal of nonlinear sciences & applications (JNSA)*, *10*(7), p. 4034. https://doi.org/10.22436/jnsa.010.07.52

[13] Ross, b. (1975). *Fractional Calculus and its applications: Proceedings of the international conference held at the university of new*. Berlin, Springer-Verlag. https://books.google.com/books/about/Fractional_Calculus_and_Its_Applications.html?id=1qkZAQAAIAAJ

[14] Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and applications of fractional differential equations*. Elsevier. https://doi.org/10.1016/S0304-0208(06)80001-0

[15] Baleanu, D., Machado, J. A. T., & Luo, A. C. (2011). *Fractional dynamics and control*. Springer Science & Business Media. https://doi.org/10.1007/978-1-4614-0457-6

[16] Baleanu, D. (2012). *Fractional calculus: Models and numerical methods*. World Scientific. https://doi.org/10.1142/10044

[17] Granas, A., & Dugundji, J. (2003). *Fixed point theory* (Vol. 14, pp. 15-16). New York: Springer. https://doi.org/10.1007/978-0-387-21593-8

[18] Deimling, K. (2013). *Nonlinear functional analysis*. Springer Science & Business Media. https://d-nb.info/850178541/04

[19] Zhang, H., Li, Y., & Yang, J. (2020). New sequential fractional differential equations with mixed-type boundary conditions. *Journal of function spaces*, *2020*(1), 6821637. https://doi.org/10.1155/2020/6821637